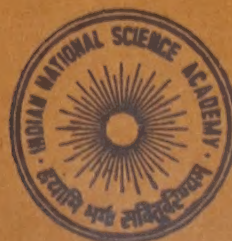


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A FIXED POINT THEOREM FOR A SEQUENCE OF MAPPINGS

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In this note we obtain a fixed point theorem for a sequence of mappings with contractive iterates.

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called a contraction mapping if there is a real number k , $0 \leq k < 1$, such that

$$d(Tx, Ty) \leq k d(x, y)$$

for all x, y in X .

The well known Banach contraction principle states that a contraction mapping of a complete metric space (X, d) into itself has a unique fixed point. In this paper we prove the existence of a common fixed point for a sequence of mappings with contractive iterates.

Theorem — Let $\{T_n\}$ be a sequence of mappings of a complete metric space (X, d) into itself such that for any two mappings T_i, T_j we have

$$(i) \quad d(T_i^m x, T_j^m y) \leq \alpha_{i,j}^{p_i} d(x, y)$$

$$(ii) \quad d(T_i^m x, T_j y) \leq \alpha_{i,j} d(x, y)$$

for some m and $0 < \alpha_{i,j} < 1$, $i, j = 1, 2, \dots$, $x, y \in X$ and is such that the series

$\sum_{l=1}^{\infty} \alpha_{l,l+1}$ is (\bar{N}, p_n) summable, i. e., $\sum_{k=1}^{\infty} \sigma_k < \infty$, where

$$\sigma_k = \frac{\sum_{\gamma=1}^k p_{\gamma} \alpha_{\gamma, \gamma+1}}{\sum_{\gamma=1}^k p_{\gamma}} .$$

and $p_n \geq 0$, $n = 2, 3, \dots$, $p_n > 0$.

Then the sequence $\{T_n\}$ has a unique common fixed point.

PROOF : Let x_0 be any point of X and $x_1 = T_1^m x_0$, $x_2 = T_2^m x_1$, Then

$$d(x_1, x_2) = d(T_1^m x_0, T_2^m x_1)$$

$$\leq \alpha_{1,2}^{p_1} d(x_0, x_1)$$

$$d(x_2, x_3) = d(T_2^m x_1, T_3^m x_2)$$

$$\leq \alpha_{2,3}^{p_2} d(x_1, x_2)$$

$$\leq \alpha_{2,3}^{p_2} \alpha_{1,2}^{p_1} d(x_0, x_1)$$

and so on. By induction we have

$$d(x_n, x_{n+1}) \leq \prod_{i=1}^n (\alpha_{i,i+1})^{p_i} d(x_0, x_1).$$

For $p > 0$, we have,

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \dots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \sum_{k=n}^{n+p-1} \left(\prod_{i=1}^k (\alpha_{i,i+1})^{p_i} \right) d(x_0, x_1)$$

$$\leq \sum_{k=n}^{n+p-1} \left(\frac{\sum_{i=1}^k p_i \alpha_{i,i+1}}{\sum_{i=1}^k p_i} \right)^{\sum_{i=1}^k p_i} d(x_0, x_1)$$

$$\leq \sum_{k=n}^{n+p-1} \left(\frac{\sum_{i=1}^k p_i s_i}{\sum_{i=1}^k p_i} \right)^{\sum_{i=1}^k p_i} d(x_0, x_1)$$

$$\leq d(x_0, x_1) \sum_{k=n}^{n+p-1} \sigma_k.$$

Since $\sum_{k=1}^{\infty} \sigma_k < \infty$ we obtain

$$d(x_n, x_{n+p}) \rightarrow 0 \text{ as } n \rightarrow \infty, p=1,2$$

Hence $\{x_n\}$ is a Cauchy sequence in X and therefore it converges to a limit, say x in X . Now, for a fixed k , we shall show that $T_k x = x$. Indeed we have for any $n = 1, 2, \dots$

$$\begin{aligned} d(x, T_k x) &\leq d(x, x_n) + d(x_n, T_k x) \\ &= d(x, x_n) + d(T_n^m x_{n-1}, T_k x) \\ &\leq d(x, x_n) + \alpha_{n,k} d(x_{n-1}, x). \end{aligned}$$

Since $\lim_n x_n = x$, $0 < \alpha_{n,k} < 1$, it follows that $x = T_k x$. To prove the uniqueness of x let if possible $x' \in X$ be such that $x' \neq x$ and $T_k x' = x'$. Then

$$\begin{aligned} d(x, x') &= d(T_i^m x, T_j^m x') \\ &\leq \alpha_{i,j}^{p_i} d(x, x') \end{aligned}$$

which is impossible since $0 < \alpha_{i,j} < 1$. Hence $x = x'$.

The theorem yields Theorem III of Chatterjea¹ when $p_i = 1$ for all i .

The hypothesis of (\bar{N}, p_n) summability makes the theorem to have wider applications. For example when $p_n = \frac{1}{n+1}$ there are series summable (\bar{N}, p_n) but not summable $(C, 1)$ (see Hardy², p. 59).

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ON AN ERROR TERM RELATED TO THE GREATEST DIVISOR OF n , WHICH IS PRIME TO k

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The object of the paper is to prove the following inequalities.

$$\lim_{x \rightarrow \infty} \frac{E_k(x)}{x} \gg \frac{k}{\sigma(k)}$$

and

$$\lim_{x \rightarrow \infty} \frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)}$$

where k is any fixed square free integer and $E_k(x)$ is the error term, which will be defined in section 2.

1. INTRODUCTION

Fix a square free integer k . We define

$$\delta_k(n) = \max \{d : d \mid n, (d, k) = 1\}.$$

Joshi and Vaidya² proved that

$$\sum_{n \leq x} \delta_k(n) = \frac{k}{2\sigma(k)} x^2 + E_k(x), \text{ with}$$

$$E_k(x) = O(x).$$

They also proved when $k = p$ (a prime)

$$\lim_{x \rightarrow \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}$$

and

$$\lim_{x \rightarrow \infty} \frac{E_p(x)}{x} = \frac{p}{p+1}.$$

It has been recently claimed by Maxsein and Herzog³, for any square free k .

$$\lim_{x \rightarrow \infty} \frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)}$$

and

$$\lim_{x \rightarrow \infty} \frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}.$$

They have applied Tauberian theorem of Hardy-Littlewood and Karamata to get an asymptotic formula for $\sum_{n \leq x} \gamma_k(n)$, where $\gamma_k(n)$ is defined by the relation $\delta_k(n) = (\gamma_k * I)(n)$ where $*$ is the Dirichlet convolution and I is the identity function.

In this paper we point out that the method of Erdős-Shapiro¹ of averaging over arithmetic progressions also yields the result.

2. DEFINITIONS AND PRELIMINARIES

We define

$$H_k(x) = \sum_{n \leq x} \frac{\delta_k(n)}{n} - \frac{xk}{\sigma(k)} \quad \dots(2.1)$$

and

$$E_k(x) = \sum_{n \leq x} \delta_k(n) - \frac{1}{2} \frac{x^2 k}{\sigma(k)}. \quad \dots(2.2)$$

We write $\delta(n)$, $E(n)$ and $H(n)$ instead of $\delta_k(n)$, $E_k(n)$ and $H_k(n)$ respectively.

Let $[x]$ denote the integral part of x and let $\lambda(x)$ denote the fractional part of x . From the definition of $\delta(n)$, it is easy to check that it is multiplicative. Now for $\sigma > 2$.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\delta(n)}{n^s} &= \prod_p \left\{ 1 + \frac{\delta(p)}{p^s} + \frac{\delta(p^2)}{p^{2s}} + \dots \right\} \\ &= \prod_{p|k} \left\{ 1 - \frac{1}{p^s} \right\}^{-1} \prod_{p \nmid k} \left\{ 1 - \frac{1}{p^{s-1}} \right\}^{-1} \\ &= \sum_{n=1}^{\infty} \frac{n}{n^s} \prod_{p|k} \left\{ \frac{1 - \frac{p}{p^s}}{1 - \frac{1}{p^s}} \right\}. \end{aligned} \quad \dots(2.3)$$

(since,

$$\begin{aligned} \delta(p^m) &= 1 \text{ if } p | k \\ &= p^m \text{ if } p \nmid k. \end{aligned}$$

Define for $\sigma > 0$,

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^{\sigma}} = \prod_{p|k} \left\{ \frac{1 - \frac{p}{p^{\sigma}}}{1 - \frac{1}{p^{\sigma}}} \right\}. \quad \dots(2.4)$$

Then we have

$$\begin{aligned} g(p^m) &= 1 - p \text{ if } p | k \\ &= 0 \text{ if } p \nmid k \end{aligned}$$

and

$$\delta(n) = \sum_{ab=n} a g(b). \quad \dots(2.5)$$

We make the following observations :

$$(I) \quad |g(n)| \leq \prod_{i=1}^r (p_i - 1), \text{ where } k = p_1 \dots p_r \text{ is the factorization of } k \text{ into primes.}$$

$$(II) \quad \sum_{n \leq x} g(n) = O((\log x)^r) = o(x).$$

$$(III) \quad \sum_{n=1}^{\infty} \frac{g(n)}{n^2} = \prod_{p|k} \left(\frac{p}{p+1} \right) = \frac{k}{\sigma(k)} \text{ from (2.4).}$$

$$\begin{aligned} (IV) \quad \sum_{n > x} \frac{g(n)}{n^2} &= \sum_{m=1}^{\infty} \sum_{2^{m-1}x \leq n \leq 2^m x} \frac{g(n)}{n^2} \\ &\leq \sum_{m=1}^{\infty} \frac{\epsilon(2^m - 2^{m-1})x}{2^{2m-2}x^2}, \epsilon \text{ small positive constant} \\ &\Rightarrow \sum_{n > x} \frac{g(n)}{n^2} = o\left(\frac{1}{x}\right). \end{aligned}$$

$$(V) \quad \sum_{n=1}^{\infty} \frac{g(n)}{n} = 0 \text{ (follows from (2.4)).}$$

$$(VI) \quad \sum_{n=1}^{\infty} \frac{|g(n)|}{n} = O(1) \text{ (follows from (2.4)).}$$

3. SOME LEMMAS

Lemma 3.1—We have

$$H(x) = - \sum_{b \leq x} \frac{g(b)}{b} \lambda\left(\frac{x}{b}\right) + o(1) = o(x).$$

PROOF: From (2.1) and (2.5) we have

$$\begin{aligned} H(x) &= \sum_{ab \leq x} \frac{ag(b)}{ab} - \frac{xk}{\sigma(k)} \\ &= \sum_{b \leq x} \frac{g(b)}{b} \left[\frac{x}{b} \right] - \frac{xk}{\sigma(k)} \\ &= x \sum_{b \leq x} \frac{g(b)}{b^2} - \sum_{b \leq x} \frac{g(b)}{b} \lambda\left(\frac{x}{b}\right) - \frac{xk}{\sigma(k)} \\ &= - \sum_{b \leq x} \frac{g(b)}{b} \lambda\left(\frac{x}{b}\right) + o(1) \text{ (by III and IV)} \\ &= O(\log x) \text{ (by VI)} \\ &= o(x). \end{aligned}$$

Lemma 3.2—We have

$$E(x) = x H(x) + o(x).$$

PROOF: From (2.2) and (2.5), we have

$$\begin{aligned} E(x) &= \sum_{b \leq x} g(b) \sum_{a \leq x/b} a - \frac{1}{2} \frac{x^2 k}{\sigma(k)} \\ &= \frac{1}{2} \sum_{b \leq x} g(b) \left\{ \left[\frac{x}{b} \right]^2 + \left[\frac{x}{b} \right] \right\} - \frac{1}{2} \frac{x^2 k}{\sigma(k)} \\ &= \frac{1}{2} \sum_{b \leq x} g(b) \left\{ \frac{x^2}{b^2} - 2 \frac{x}{b} \lambda\left(\frac{x}{b}\right) + \lambda^2\left(\frac{x}{b}\right) \right\} \\ &\quad + \frac{1}{2} \sum_{b \leq x} g(b) \left\{ \frac{x}{b} - \lambda\left(\frac{x}{b}\right) \right\} - \frac{1}{2} x^2 \frac{k}{\sigma(k)} \\ &= -x \sum_{b \leq x} \frac{g(b)}{b} \lambda\left(\frac{x}{b}\right) + o(x). \end{aligned} \quad \dots(3.2.1)$$

Since $0 \leq \lambda^2 \left(\frac{x}{b} \right) < 1$, we have

$$\sum_{b \leq x} g(b) \lambda^2 \left(\frac{x}{b} \right) = O \left(\sum_{b \leq x} |g(b)| \right) = o(x).$$

From (3.2.1) and Lemma 3.1, the lemma follows.

Lemma 3.3—Let x be an integer. Then

$$\sum_{n \leq x} H(n) = \frac{kx}{2\sigma(k)} + o(x).$$

PROOF : From (2.1)

$$\begin{aligned} \sum_{n \leq x} H(n) &= \sum_{n \leq x} \left\{ \sum_{m \leq n} \frac{\delta(m)}{m} - \frac{nk}{\sigma(k)} \right\} \\ &= \sum_{n \leq x} (x+1-n) \frac{\delta(n)}{n} - \frac{k}{\sigma(k)} \frac{x(x+1)}{2} \\ &= (x+1) \sum_{n \leq x} \frac{\delta(n)}{n} - \sum_{n \leq x} \delta(n) - \frac{k}{2\sigma(k)} x(x+1) \\ &= (x+1) \left(H(x) + \frac{xk}{\sigma(k)} \right) - \left(E(x) + \frac{kx^2}{2\sigma(k)} \right) \\ &\quad - \frac{k}{2\sigma(k)} x(x+1) \\ &= (x+1) H(x) - E(x) + \frac{kx}{2\sigma(k)} \\ &= H(x) + \frac{kx}{2\sigma(k)} + o(x), \text{ (by Lemma (3.2))} \\ &= \frac{kx}{2\sigma(k)} + o(x), \text{ (by Lemma (3.1)).} \end{aligned}$$

Lemma 3.4—Let $A = (2 \cdot P^n)$, $n \geq 2$ where $P = 2$ if $2 \mid k$,
 $=$ odd prime dividing k if $2 \nmid k$.

If β is an integer with $0 < \beta < A$, then

$$\sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \frac{\delta(m)}{m} = \frac{z}{A} \cdot \frac{k}{\sigma(k)} \left(\frac{1+P}{P} \right) \sum_{\substack{t \mid (A, \beta) \\ P^n \nmid t}} \frac{g(t)}{t}$$

$$+ O\left(\frac{1}{P^n}\right) \frac{z}{A} + O(\log z).$$

PROOF :
$$\sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \frac{\delta(m)}{m} = \sum_{\substack{m \leq z \\ m \equiv \beta(A)}} \sum_{d|m} \frac{g(d)}{d}$$

$$= \sum_{d \leq z} \frac{g(d)}{d} \sum_{\substack{m \leq z \\ m \equiv \beta(A) \\ m \equiv d(A)}} 1$$

$$= \sum_{\substack{d \leq z \\ (d,A) | \beta}} \frac{g(d)}{d} \left(\frac{z}{[d,A]} + O(1) \right).$$

(The congruences $m \equiv \beta(A)$, $m \equiv 0(d)$ are simultaneously solvable iff $(d, A) | \beta$ and in that case there is a unique solution modulo $[d, A]$)

$$= \frac{z}{A} \sum_{\substack{d \leq z \\ (d,A) | \beta}} \frac{g(d)}{d^2} (d, A) + O\left(\sum_{d \leq z} \frac{g(d)}{d}\right)$$

$$(\because (d, A)[d, A] = dA)$$

$$= \frac{z}{A} \sum_{t | (A, \beta)} t \sum_{\substack{d \leq z \\ (d,A)=t}} \frac{g(d)}{d^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{t | (A, \beta)} \frac{1}{t} \sum_{\substack{m \leq z/t \\ (mA/t)=1}} \frac{g(mt)}{m^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{\substack{t | (A, \beta) \\ P^n \nmid t}} \frac{g(t)}{t} \sum_{\substack{m \leq z/t \\ (m,P)=1}} \frac{g(m)}{m^2} + O\left(\frac{1}{P^n}\right) \frac{z}{A} + O(\log z).$$

(\because The first term correspond to $(m, t) = 1$)

$$= \frac{z}{A} \sum_{\substack{t | (A, \beta) \\ P^n \nmid t}} \frac{g(t)}{t} \left\{ \sum_{\substack{m=1 \\ (P,n)=1}}^{\infty} \frac{g(m)}{m^2} - \sum_{\substack{m \geq z/t \\ (P,m)=1}} \frac{g(m)}{m^2} \right\}$$

$$+ O\left(\frac{1}{P^n}\right) \frac{z}{A} + O(\log z)$$

$$= \frac{z}{A} \frac{k}{\sigma(k)} \left(\frac{1+P}{P} \right) \sum_{\substack{t | (A, \beta) \\ P^n \nmid t}} \frac{g(t)}{t} + O\left(\frac{1}{P^n}\right) \frac{z}{A} + O(\log z).$$

Since, by putting $X = \frac{z}{t}$, we have

$$\begin{aligned} \sum_{\substack{m \leq z/t \\ (P,m)=1}} \frac{g(m)}{m^2} &= \sum_{\substack{m=1 \\ (P,m)=1}}^{\infty} \frac{g(m)}{m^2} - \sum_{\substack{m > z/t \\ (P,m)=1}} \frac{g(m)}{m^2} \\ &= \prod_{\substack{p|k \\ p \neq P}} \left(\frac{1 - \frac{p}{p^2}}{1 - \frac{1}{p^2}} \right) + O \left(\sum_U \sum_{U \leq m \leq 2U} \frac{|g(m)|}{m^2} \right) \\ &\quad \text{where } U = 2^n X. \\ &= \frac{k}{\sigma(k)} \frac{(1+P)}{P} + O \left(\sum_U \frac{(\log U)^r}{U^2} \right). \quad \dots(3.4.1) \end{aligned}$$

$$\begin{aligned} \sum_U \frac{(\log U)^r}{U^2} &= O \left(\sum_{n=0}^{\infty} \frac{(\log X + n \log 2)^r}{2^{2n} X^2} \right) \\ &= O \left(\sum_{n > \log X} \frac{n^r}{2^{2n} X^2} + \sum_{n < \log X} \frac{(\log X)^r}{2^{2n} X^2} \right) \\ &= O \left(\frac{(\log X)^2}{X^2} \right). \quad \dots(3.4.2) \end{aligned}$$

(3.4.1) and (3.4.2) prove the lemma.

Lemma 3.5—For $0 < B < A$, and integral x , we have

$$\begin{aligned} \sum_{l \leq x} H(A l - B) &= \frac{x k}{\sigma(k)} \left\{ B - \frac{A}{2} + \left(1 + \frac{1}{P} \right) \sum_{\substack{t | (A, -B) \\ P^n \nmid t}} \frac{g(t)}{t} \right. \\ &\quad \left. - \left(1 + \frac{1}{P} \right) \frac{1}{A} \sum_{a=0}^{A-1} (B-a) \sum_{\substack{t | (A, a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \right\} \\ &\quad + O \left(\frac{1}{P^n} \right) \frac{x}{A} \sum_{a=0}^{A-1} (B-a) + o(x). \end{aligned}$$

PROOF :

$$\sum_{l \leq x} H(A l - B) = \sum_{l \leq x} \left(\sum_{m \leq A l - B} \frac{\delta(m)}{m} - \frac{(A l - B) k}{\sigma(k)} \right)$$

(equation continued on p. 837)

$$\begin{aligned}
&= \sum_{m \leq Ax-B} \frac{\delta(m)}{m} \left\{ x - \left[\frac{m+B}{A} \right] \right\} + \sum_{\substack{m \leq Ax-B \\ m \equiv -B(A)}} \frac{\delta(m)}{m} \\
&\quad - \sum_{l \leq x} (Al-B) \frac{k}{\sigma(k)}. \quad \dots(3.5.1)
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{m \leq Ax-B} \frac{\delta(m)}{m} \left\{ x - \left[\frac{m+B}{A} \right] \right\} \\
&= \left\{ x \sum_{m \leq Ax-B} \frac{\delta(m)}{m} - \frac{1}{A} \sum_{m \leq Ax-B} \delta(m) \right\} - \sum_{a=0}^{A-1} \frac{B-a}{A} \\
&\quad \times \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m} \\
&\quad - \frac{1}{A} \left\{ (Ax-B+1) \left(\sum_{m \leq Ax-B} \frac{\delta(m)}{m} - \frac{(Ax-B)k}{\sigma(k)} \right) \right. \\
&\quad \left. - \left(\sum_{m \leq Ax-B} \delta(m) - \frac{1}{2} (Ax-B)^2 \frac{k}{\sigma(k)} \right) + \frac{(Ax-B)k}{2\sigma(k)} \right\} \\
&\quad + \frac{(B-1)}{A} \sum_{m \leq Ax-B} \frac{\delta(m)}{m} + \frac{1}{2A} \{ (Ax-B) + (Ax-B)^2 \} \\
&\quad \times \frac{k}{\sigma(k)} - \sum_{a=0}^{A-1} \frac{(B-a)}{A} \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m} \\
&= \frac{1}{A} \{ (Ax-B+1) H(Ax-B) - E(Ax-B) + \frac{(Ax-B)k}{2\sigma(k)} \} \\
&\quad + \frac{(B-1)}{A} \left\{ \sum_{m \leq Ax-B} \frac{\delta(m)}{m} - \frac{(Ax-B)k}{\sigma(k)} \right\} \\
&\quad + \frac{k}{2A\sigma(k)} \{ 2Ax-B - 2B^2 - 2Ax + 2B + Ax-B \\
&\quad + A^2x^2 + B^2 - 2Ax-B \} \\
&\quad - \sum_{a=0}^{A-1} \frac{(B-a)}{A} \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m}
\end{aligned}$$

(equation continued on p. 838)

$$\begin{aligned}
&= \frac{1}{A} \sum_{m \leq Ax-B} H(m) + \frac{(B-1)}{A} \left\{ \sum_{m \leq Ax-B} \frac{\delta(m)}{m} \frac{(Ax-B)k}{\sigma(k)} \right\} \\
&\quad + \frac{k}{2A\sigma(k)} \{A^2x^2 - B^2 - Ax + B\} - \sum_{a=0}^{A-1} \frac{(B-a)}{A} \\
&\quad \times \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m} = \frac{1}{A} \sum_{m \leq Ax-B} H(m) + \frac{Ax^2}{2} \frac{k}{\sigma(k)} \\
&\quad - \frac{xk}{2\sigma(k)} - \sum_{a=0}^{A-1} \frac{(B-a)}{A} \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m} \\
&\quad + o(x). \tag{3.5.2}
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{a=0}^{A-1} \frac{(B-a)}{A} \sum_{\substack{m \leq Ax-B \\ m+B \equiv a(A)}} \frac{\delta(m)}{m} \\
&= \sum_{a=0}^{A-1} \frac{(B-a)}{A} \left\{ \frac{Ax-B}{A} \frac{k}{\sigma(k)} \left(1 + \frac{1}{P}\right) \sum_{\substack{t \mid (A, a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \right. \\
&\quad \left. + O\left(\frac{1}{P^n}\right) \cdot \frac{Ax-B}{A} + O(\log x) \right\} \text{ (by Lemma 3.4)} \\
&= \left(1 + \frac{1}{P}\right) \frac{xk}{\sigma(k)} \frac{1}{A} \sum_{a=0}^{A-1} (B-a) \sum_{\substack{t \mid (A, a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \\
&\quad + O\left(\frac{1}{P^n}\right) x \sum_{a=0}^{A-1} \frac{(B-a)}{A} + O(\log x). \tag{3.5.3}
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{\substack{m \leq Ax-B \\ m \equiv -B(A)}} \frac{\delta(m)}{m} = \frac{(Ax-B)k}{A\sigma(k)} \left(1 + \frac{1}{P}\right) \sum_{\substack{t \mid (A-B) \\ P^n \nmid t}} \frac{g(t)}{t} \\
&\quad + O\left(\frac{1}{P^n}\right) \frac{Ax-B}{A} + O(\log x). \tag{3.5.4}
\end{aligned}$$

3.5.1, 3.5.2, 3.5.3, 3.5.4 prove the Lemma.

4. MAIN THEOREMS

Theorem 4.1—We have the following inequality

$$\lim_{x \rightarrow \infty} \frac{E(x)}{x} \geq \frac{k}{\sigma(k)}.$$

PROOF: By Lemma 3.5, we have for integral x ,

$$\begin{aligned} \sum_{l \leq x} H(A| - B) &= \frac{xk}{\sigma(k)} \left\{ B - \frac{A}{2} + \left(1 + \frac{1}{P}\right) \sum_{\substack{t | (A, B) \\ P^n \nmid t}} \frac{g(t)}{t} \right. \\ &\quad \left. - \left(1 + \frac{1}{P}\right) \frac{1}{A} \sum_{a=0}^{A-1} (B-a) \sum_{\substack{t | (A, a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \right\} \\ &\quad + O\left(\frac{1}{P^n}\right) \frac{x}{A} \sum_{a=0}^{A-1} (B-a) + o(x). \quad \dots (4.1.1) \end{aligned}$$

We have $A = 2P^n$. Take $B = P^n + 1$, then

$$\begin{aligned} B - \frac{A}{2} + \left(1 + \frac{1}{P}\right) \sum_{\substack{t | (A, B) \\ P^n \nmid t}} \frac{g(t)}{t} - \left(1 + \frac{1}{P}\right) \frac{1}{A} \sum_{a=0}^{A-1} (B-a) \sum_{\substack{t | (A, a-B) \\ P^n \nmid t}} \frac{g(t)}{t} \\ = 1 + \left(1 + \frac{1}{P}\right) - \left(1 + \frac{1}{P}\right) \frac{1}{2P^n} \left(\sum_{r=1}^{P^n+1} r - \sum_{r=1}^{P^n-2} r \right) \\ \times \sum_{\substack{t | (A, -r) \\ P^n \nmid t}} \frac{g(t)}{t}. \quad \dots (4.1.2) \end{aligned}$$

(Since, when $a = 0$ to $A - 1$, $B - a = B, B - 1, \dots, B - (A - 1)$
 $= P^n + 1, P^n, \dots, (P^n - 2).$)

Now,

$$\left(1 + \frac{1}{P}\right) \frac{1}{2P^n} \left(\sum_{r=1}^{P^n+1} r - \sum_{r=1}^{P^n-2} r \right) \sum_{\substack{t | (2A, -r) \\ P^n \nmid t}} \frac{g(t)}{t}$$

(equation continued on p. 840)

$$\begin{aligned}
&= \left(1 + \frac{1}{P}\right) \frac{1}{2P^n} ((P^n + 1) \cdot 1 + P^n \sum_{\substack{t \mid (2P^n - P^n) \\ P^n \nmid t}} \frac{g(t)}{t} + P^n - 1)) \\
&= \left(1 + \frac{1}{P}\right) \frac{1}{2P^n} \left(2P^n + P^n \sum_{t \mid P^{n-1}} \frac{g(t)}{t}\right) \\
&= \left(1 + \frac{1}{P}\right) \frac{1}{2} \left(2 + \left(1 + (1 - P) \left(\frac{1}{P} + \frac{1}{P^2} + \dots + \frac{1}{P^{n-1}}\right)\right)\right) = \left(1 + \frac{1}{P}\right) - \frac{1}{2} \left(1 + \frac{1}{P}\right) \left(\frac{1}{P^{n-1}}\right) \\
&= \left(1 + \frac{1}{P}\right) + O\left(\frac{1}{P^{n-1}}\right). \quad \dots(4.1.3)
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{a=0}^{A-1} (B - a) &= B + (B - 1) + \dots + (B - A + 1) \\
&= (2 - P^n) + (3 - P^n) + \dots + P^n + (P^n + 1) \\
&= \frac{1}{2} \{(P^n + 1)(P^n + 2) + (1 - P^n)(P^n - 2)\} \\
&= 3P^n \\
\therefore O\left(\frac{1}{P^n}\right) \frac{x}{A} \sum_{a=0}^{A-1} (B - a) &= O\left(\frac{1}{P^n}\right) \frac{x}{2P^n} (3P^n) \\
&= O\left(\frac{1}{P^n}\right) x. \quad \dots(4.1.4)
\end{aligned}$$

\therefore From (4.1.1), (4.1.2), (4.1.3) and (4.1.4) we have

$$\sum_{l \leq x} H(A^l - B) = \frac{xk}{\sigma(k)} \left\{1 + O\left(\frac{1}{P^{n-1}}\right)\right\} + O\left(\frac{1}{P^n}\right)x + o(x).$$

This is true for all n , n -arbitrarily large.

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{E(x)}{x} \geq \frac{k}{\sigma(k)}.$$

Theorem 4.2—We have the following inequality :

$$\lim_{x \rightarrow \infty} \frac{E(x)}{x} \leq -\frac{k}{\sigma(k)}.$$

PROOF : By choosing $B = P^n$ in Lemma 3.5, it is easy to see that

$$\lim_{x \rightarrow \infty} H(x) \leq 0 \text{ (for integral } x\text{).}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{E(x)}{x} \leq 0.$$

$\dots(4.2.1)$

Now

$$E(x) - E([x]) = \frac{k}{2\sigma(k)} \{-2[x]\lambda(x) - \lambda^2(x)\}$$

which shows that $E(x)$ decreases continuously in any open interval $(m, m+1)$.

$$\frac{E(x)}{x} = \frac{E([x])}{x} - \frac{k}{\sigma(k)} \frac{[x]}{x} \lambda(x) - \frac{k \lambda^2(x)}{2x \sigma(k)}.$$

As $\lambda(x) \rightarrow 1$ from the left,

$$\begin{aligned} \frac{E(x)}{x} &\rightarrow \frac{E([x])}{x} - \frac{k}{\sigma(k)} \frac{[x]}{x} - \frac{k}{2x \sigma(k)} \\ \text{ie., } \frac{E(x)}{x} &\rightarrow \frac{E([x])}{x} - \frac{k}{\sigma(k)} \left(1 - \frac{1}{x}\right) - \frac{k}{2x \sigma(k)} \\ \text{ie., } \lim_{\lambda(x) \rightarrow 1} \frac{E(x)}{x} &= \frac{E([x])}{x} - \frac{k}{\sigma(k)} + \frac{k}{2x \sigma(k)}. \end{aligned} \quad \dots(4.2.2)$$

From (4.2.1), we have

$$\lim_{x \rightarrow \infty} \frac{E(x)}{x} \leq -\frac{k}{\sigma(k)}.$$

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A NOTE ON N -GROUPS

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For an ideal I of an N -group G in which every essential N -subgroup is strictly essential the conditions I is uniform and I is strictly uniform are equivalent. As a consequence we obtain a few results.

1. INTRODUCTION

Throughout this paper a near-ring will mean a zero symmetric right near-ring. Moreover N stands for a near-ring, G stands for an N -group and H stands for an ideal of G . The ideal generated by a subset X is denoted by (X) and we write (h) for the ideal generated by a single element ' h '. We shall utilize the standard notations and definitions as in Pilz³.

An ideal H is said to be 'essential in an ideal' K of G if (i) $H \subseteq K$ and (ii) $H \cap L = (0)$, $L \subseteq K$, L is an ideal of G , imply $L = (0)$. Besides that, H is said to be essential if it is essential in G . Intersection of a finite number of essential ideals is essential and every ideal containing an essential ideal is essential. One can easily verify that if H_1, H_2, K_1, K_2 are ideals of G such that the sum $K_1 + K_2$ is direct and H_i is essential in K_i for $i = 1, 2$, then $H_1 + H_2$ is essential in $K_1 + K_2$. Moreover H is said to be 'uniform' if for each pair of ideals K_1 and K_2 of G such that $K_1 \cap K_2 = (0)$ $K_1 \subseteq H$ and $K_2 \subseteq H$, implies that $K_1 = (0)$ or $K_2 = (0)$. On the other hand, H is said to have 'finite Goldie dimension' if it does not contain an infinite number of nonzero ideals of G whose sum is direct. As in ring theory we have the following conclusions:

(i) H has finite Goldie dimension if and only if for any sequence $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$ of ideals of G with each $H_i \subseteq H$ there exists an integer k such that H_i is essential in H_{i+1} for $i \geq k$;

(ii) If H has finite Goldie dimension then H contains a uniform ideal and

(iii) If G has finite Goldie dimension then every ideal of G has finite Goldie dimension.

The ideal H is said to be 'minimal' if H is minimal in the set of nonzero ideals of G and the N -group. G is said to be 'completely reducible' if G is a direct sum of minimal ideals.

Notation : 0.1 : For any nonempty subset A of G we write

$$A^* = \{g + x - g/x \in A, g \in G\}$$

$$A^0 = \{x - y \mid x, y \in A\}$$

$$A^+ = \{n(g + x) - ng/x \in A, g \in G, n \in N\}.$$

Let X be a nonempty subset of G and write $X_0 = X$, and $X_{i+1} = X_i^* \cup X_i^+ \cup X_i^0$ for all integers $i \geq 0$. Then $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ and clearly $\bigcup_{i=0}^{\infty} X_i$ is the ideal generated by X .

2. UNIFORM IDEALS

Before proving our main theorem (that is, Theorem 3.1), we prove the following two lemmas.

Lemma 2.1—Suppose H and K are two ideals of G such that $H \cap K = (0)$ and suppose $a \in H$, and $b \in K$. Then for any $x \in (a)$, there exists $y \in (b)$ such that $x + y \in (a + b)$.

PROOF : Write $A = \{a\}$ and $B = \{b\}$ and $C = \{a + b\}$. Following the notation 0.1, $(a) = \bigcup_{i=0}^{\infty} A_i$; $(b) = \bigcup_{i=0}^{\infty} B_i$; $(a + b) = \bigcup_{i=0}^{\infty} C_i$. For each positive integer $k \geq 0$, suppose $P(k)$ is the statement: $x \in A_k$ implies there is a $y \in B_k$ such that $x + y \in C_k$.

Clearly $P(0)$ is true. Suppose $P(k-1)$ is true for some positive integer k . Let $x \in A_k$. $x \in A_k \Rightarrow x = x_1 - x_2$ or $x = g + x_1 - g$ or $x = n(g + x_1) - ng$ for some $x_1, x_2 \in A_{k-1}$, $g \in G$ and $n \in N$. Let $y_1, y_2 \in B_{k-1}$ such that $x_1 + y_1, x_2 + y_2 \in C_{k-1}$. If $x = x_1 - x_2$ then $y = y_1 - y_2$; if $x = g + x_1 - g$, then $y = g + y_1 - g$; if $x = n(g + x_1) - ng$, then $y = n(g + x_1 + y_1) - n(g + x_1)$ will satisfy $x + y \in C_k$. Thus $P(k)$ is true. Hence the proof of the lemma.

Lemma 2.2—Suppose H, I, J, K are ideals of G such that $I \cap J = (0)$ and H is essential in I and K is essential in J . Then $H + K$ is essential in $I + J$.

PROOF : Write $A = H + J$ and $B = I + K$. We now show A is essential in $I + J$. Let $0 \neq a \in (I + J)$. $a = x + s$ for some $x \in I$ and $s \in J$. If $x = 0$, then $a \in A$ and hence $(a) \cap A \neq (0)$. If $x \neq 0$, then since H is essential in I , there exists a nonzero element x' in $(x) \cap H$. By Lemma 2.1, there exists y' in (s) such that $x' + y' \in (a)$. Therefore $x' + y'$ is in $(a) \cap A$ and hence A is essential in $I + J$. Similarly one can show B is essential in $I + J$. Thus $H + K = A \cap B$ is essential in $I + J$.

Corollary 2.3—Suppose $H_1, H_2, \dots, H_t, K_1, K_2, \dots, K_t$ are ideals of G such that the sum $K_1 + K_2 + \dots + K_t$ is direct and $H_i \subseteq K_i$ for $i = 1, 2, \dots, t$. Then the following are equivalent.

- (i) H_i is essential in K_i for $i = 1, 2, \dots, t$.

- (ii) $H_1 + H_2 + \dots + H_t$ is essential in $K_1 + K_2 + \dots + K_t$.

Using this corollary, the proof of the following theorem is similar to that of a corresponding Theorem.

Theorem 2.4—For an N -group G , the following are equivalent.

- (i) G has finite Goldie dimension
 (ii) There exist uniform ideals U_1, U_2, \dots, U_t in G such that the sum $U_1 + U_2 + \dots + U_t$ is direct and essential in G .

Note : If G has finite Goldie dimension then the integer t determined in the above theorem is called the dimension of G and it is denoted by $\dim G$.

Corollary 2.5—Suppose $\dim G = k$. Then (i) the number of summands in any decomposition of H as the direct sum of nonzero ideals of G is at most k and (ii) H is essential if and only if H contains direct sum of k uniform ideals.

Suppose G has finite Goldie dimension. Then H has finite Goldie dimension and hence there exist k uniform ideals in G whose sum is direct and essential in H . This number k is called the dimension of H and is denoted by $\dim H$. Clearly a nonzero ideal U of G is uniform if and only if $\dim U = 1$.

For an N -group G having finite Goldie dimension, the following holds.

- (i) H is essential if and only if $\dim H = \dim G$.
 (ii) If H, K are two ideals of G such that $H \cap K = (0)$, then $\dim (H + K) = \dim H + \dim K$.
 (iii) If $\dim H < \dim G$, then there exist uniform ideals U_1, U_2, \dots, U_k of G such that the sum $H + U_1 + U_2 + \dots + U_k$ is direct and essential in G . Moreover, $k = \dim G - \dim H$.
 (iv) If G is completely reducible, then an ideal of G is minimal if and only if it is uniform.

3. STRICTLY UNIFORM IDEALS

An ideal (N -subgroup) K of G is said to be strictly uniform if for any two N -subgroups A, B of G , $A \subseteq K, B \subseteq K, A \cap B = (0)$ implies $A = (0)$ or $B = (0)$. An N -subgroup K is said to be essential (strictly essential) if $H \cap A = (0)$, A is an ideal (N -subgroup) of G imply $A = (0)$. These definitions are due to Oswald³.

Every strictly uniform ideal is uniform and every strictly essential N -subgroup is essential. Every N -subgroup of G is essential if and only if every ideal of G is strictly essential.

Theorem 3.1—Suppose G is an N -group in which every essential N -subgroup is strictly essential. Then every uniform ideal of G is strictly uniform.

PROOF : Suppose U is not a strictly uniform ideal. Let A, B be two nonzero N -subgroups contained in U such that $A \cap B = (0)$. Suppose C is an ideal maximal with respect to $C \cap A = (0)$. Then $A + C$ is essential N -subgroup of G and hence it is strictly essential. Then there exists a non-zero element b in $B \cap (A + C)$. Let $a \in A, c \in C$ be such that $a + c = b$. Since $A \cap B = (0)$ the element $c = (-a) + b$ is nonzero and it is in $C_1 = U \cap C$. Suppose K is an ideal of G such that $K \cap C_1 = (0)$ and $K + C_1$ is essential. Then $K + C_1$ is strictly essential and $A \cap (K + C_1) \neq (0)$. As above one can verify $D = K \cap U \neq (0)$. Thus C and D are nonzero ideals which are contained in U such that $C \cap D = (0)$. Therefore U is not uniform. This completes the proof.

Corollary 3.2—Suppose G is an N -group in which every essential N -group is strictly essential. Then the following are equivalent.

- (i) G has finite Goldie dimension.
- (ii) There exist a finite number of uniform ideals in G whose sum is direct and essential in G .
- (iii) There exist a finite number of strictly uniform ideals in G whose sum is direct and essential in G . We close with the following example.

Example 3.3—Suppose G is the symmetric group (written additively) on three elements. Then G can be considered as a Z -group where Z is the ring of integers. If P is the alternating subgroup of G then because P is the only proper ideal of G , it follows that G is uniform. However G has three Z -subgroups each having two elements. If A, B are two of such Z -subgroups then $A \cap B = (0)$. Thus G is not strictly uniform.

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THE INTRINSIC GAUSS, CODAZZI AND RICCI EQUATIONS FOR THE BERWALD CONNECTION IN A FINSLER HYPERSURFACE

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The intrinsic Gauss-Codazzi equations for the Cartan connection have been obtained by I. Čomic'. With respect to the Berwald connection, R. S. Sinha obtained the equations for the intrinsic h -curvature tensor only. In the present paper, we will derive the complete system of the Gauss, Codazzi, second Codazzi and Ricci equations for the intrinsic Berwald connection in a Finsler hypersurface.

INTRODUCTION

The Gauss-Codazzi equations in the theory of Finsler hypersurfaces are well-known for the induced connections^{1,8}. Recently Matsumoto⁵ developed a systematic research of the hypersurface theory based on his axiomatic theory of Finsler connections, and obtained the Gauss-Codazzi equations for an induced general connection. On the other hand, Rund⁸ and Čomic'² have obtained the intrinsic Gauss-Codazzi equations for the Rund and Cartan connections, respectively. Sinha¹⁰ has obtained the equations for the intrinsic h -curvature tensor of the Berwald connection.

The purpose of the present paper is to derive the complete system of the Gauss, Codazzi, second Codazzi and Ricci equations for the intrinsic Berwald connection in the similar way to Matsumoto⁵. As for geometrical applications of those equations, we shall refer the readers to T. Yamada's paper¹¹. The similar applications of the intrinsic Gauss-Codazzi equations for the Cartan connection are seen in the paper Fukui.³

The notation used here is that of Matsumoto⁵ and Fukui³.

2. PRELIMINARIES

Let $F^n = (M^n, L(x, y))$ be an n -dimensional Finsler space, where M^n is a manifold and $L(x, y)$ is its fundamental function. $L(x, y)$ is assumed to be positively homogeneous of degree one in $y = (y^i)$. (Throughout the present paper, Latin indices take values $1, \dots, n$). The metric tensor and others are defined by

$$g_{ij} = (\partial_i \partial_j L^2)/2, \quad C_{ijk} = (\partial_k g_{ij})/2, \quad l_i = \partial_i L, \quad h_{ij} = g_{ij} - l_i l_j, \quad \dots (2.1)$$

where $\dot{\partial}_i = \partial/\partial y^i$. We are concerned with the Berwald connection $B\Gamma = (G_{ij}^i, G_j^i, 0)$ on F^n . The connection coefficients G_{jk}^i and G_j^i are determined from the fundamental function L by $G_{jk}^i = \dot{\partial}_k G_j^i$, $G_j^i = \dot{\partial}_j G^i$, $G^i = g^{ij} G_j$ and

$$2G_j = y^i \dot{\partial}_j \partial_i (L^2/2) - \partial_j (L^2/2) \quad \dots(2.2)$$

where $\partial_i = \partial/\partial x^i$. The δ -differentiation is given by

$$\delta_j = \partial_j - G_j^m \partial_m \quad \dots(2.3)$$

and the covariant differentiations are written as

$$X_{;j}^i = \delta_j X^i + G_{jk}^i X^k, \quad X_{;i}^j = \dot{\partial}_j X^i. \quad \dots(2.4)$$

In particular, we have

$$g_{ij;k} = -2C_{ijk}|0, \quad g_{ij;k} = 2C_{ijk} \quad \dots(2.5)$$

where the vertical bar denotes Cartan's h -covariant differentiation and 0 denotes the contraction with the element of support y . The Ricci formulas of this connection are written in the form

$$\begin{aligned} X_{;j;k}^i - X_{;k;j}^i &= X^m H_{mjk}^i - X_{;m}^i R_{jk}^m, \quad X_{;j;k}^i - X_{;k;j}^i = X^m G_{mjk}^i, \\ X_{;j;k}^i - X_{;k;j}^i &= 0 \end{aligned} \quad \dots(2.6)$$

where the tensors H_{hjk}^i , R_{jk}^i and G_{hjk}^i are given by

$$\begin{aligned} R_{jk}^i &= \delta_k G_j^i - \delta_j G_k^i, \quad H_{hjk}^i = \partial_h R_{jk}^i = \delta_k G_{hj}^i + G_{hj}^m G_{mk}^i - j | k, \\ G_{hjk}^i &= \dot{\partial}_h G_{jk}^i \end{aligned} \quad \dots(2.7)$$

where $j | k$ means the sum of terms obtained by interchanging j and k of all the preceding terms.

A hypersurface M^{n-1} of the manifold M^n may be represented parametrically in the form $x^i = x^i(u^\alpha)$ (all the Greek indices run from 1 to $n-1$). In the following we employ the notations

$$B_\alpha^i = \partial x^i / \partial u^\alpha, \quad B_{\alpha\beta}^i = \partial_\beta B_\alpha^i, \quad B_{\alpha\beta \dots \gamma}^{ij \dots k} = B_\alpha^i B_\beta^j \dots B_\gamma^k,$$

where the matrix β_α^i is assumed to be of rank $n-1$. Let y^i and v^α be components of the supporting element of M^{n-1} with respect to x^i and u^α . Then we may write

$$y^i = B_{\alpha}^i v^{\alpha}. \quad \dots(2.8)$$

The function $L(u, v) = L(x(u), y(u, v))$ gives rise to a fundamental function of a Finsler space $F^{n-1} = (M^{n-1}, L(u, v))$. Taking notice of

$$\partial_{\alpha} = B_{\alpha}^i \partial_i + B_{0\alpha}^i \dot{\partial}_i, \dot{\partial}_{\alpha} = B_{\alpha}^i \dot{\partial}_i \quad \dots(2.9)$$

we have

$$l_{\alpha} = B_{\alpha}^i l_i, g_{\alpha\beta} = g_{ij} B_{\alpha}^i B_{\beta}^j, C_{\alpha\beta\gamma} = C_{ijk} B_{\alpha}^i B_{\beta}^j B_{\gamma}^k, h_{\alpha\beta} = h_{ij} B_{\alpha}^i B_{\beta}^j. \quad \dots(2.10)$$

The unit normal $B^i(u, v)$ is defined by

$$g_{ij} B_{\alpha}^i B^j = 0, g_{ij} B^i B^j = 1. \quad \dots(2.11)$$

Further, using $B_i^{\alpha} = g^{\alpha\beta} g_{ij} B_{\beta}^j$ and $B_i = g_{ij} B^j$ we have

$$B_{\alpha}^i B_i^{\beta} = \delta_{\alpha}^{\beta}, B_{\alpha}^i B_i = 0, B^i B_i^{\alpha} = 0, B^i B_i = 1, \\ B_{\alpha}^i B_j^{\alpha} + B^i B_j = \delta_j^i. \quad \dots(2.12)$$

Defining $M_{\alpha\beta}$, M_{α} and M by

$$M_{\alpha\beta} = C_{ijk} B_{\alpha}^i B_{\beta}^j B^k, M_{\alpha} = C_{ijk} B_{\alpha}^i B^j B^k, \\ M = C_{ijk} B^i B^j B^k. \quad \dots(2.13)$$

then we have

$$\dot{\partial}_{\beta} B_{\alpha}^i = 0, \dot{\partial}_{\beta} B^i = -2M_{\beta}^{\alpha} B_{\alpha}^i - M_{\beta} B^i, \\ \dot{\partial}_{\beta} B_i^{\alpha} = 2M_{\beta}^{\alpha} B_i, \\ \dot{\partial}_{\beta} B_i = M_{\beta} B_i. \quad \dots(2.14)$$

Finally we could show easily

$$\dot{\partial}_{\gamma} M_{\beta}^{\alpha} - \dot{\partial}_{\beta} M_{\gamma}^{\alpha} + M_{\beta}^{\alpha} M_{\gamma} - M_{\gamma}^{\alpha} M_{\beta} = 0. \quad \dots(2.15)$$

3. THE INTRINSIC BERWALD CONNECTION

We are concerned with the Berwald connection $\beta\Gamma(G_{\beta\gamma}^{\alpha}, G_{\beta}^{\alpha}, 0)$ which is determined by the induced fundamental function $L(u, v)$ of F^{n-1} . At first, from (2.2) and (2.9) we get

$$2G^{\alpha} = B_i^{\alpha} (B_{00}^i + 2G^i). \quad \dots(3.1)$$

Differentiating this by v^β , we have

$$G_\beta^\alpha = M_\beta^\alpha B_l (B_{00}^l + 2G^l) + B_l^\alpha (B_{0\beta}^l + G_l^j B_\beta^j). \quad \dots(3.2)$$

If we put

$$N_\beta^\alpha = B_l^\alpha (B_{0\beta}^l + G_l^j B_\beta^j), \quad H_\beta = B_l (B_{0\beta}^l + G_l^j B_\beta^j). \quad \dots(3.3)$$

we may write (3.2) as

$$G_\beta^\alpha = N_\beta^\alpha + H_0 M_\beta^\alpha \quad \dots(3.4)$$

where $H_0 = N_\alpha v^\alpha$ and H_0 satisfies

$$\dot{\partial}_\gamma H_0 = 2H_\gamma + H_0 M_\gamma. \quad \dots(3.5)$$

Differentiating (3.4) further and using (3.3), (3.5) we have

$$\begin{aligned} G_{\beta\gamma}^\alpha &= F_{\beta\gamma}^\alpha + H_0 (\dot{\partial}_\gamma M_\beta^\alpha + M_\gamma M_\beta^\alpha) \\ &\quad + 2(H_\beta M_\gamma^\alpha + H_\gamma M_\beta^\alpha) \end{aligned} \quad \dots(3.6)$$

where we put

$$F_{\beta\gamma}^\alpha = B_l^\alpha (B_{\beta\gamma}^l + G_{l,k}^j B_\beta^j B_\gamma^k). \quad \dots(3.7)$$

If we put

$$H_{\beta\gamma} = B_l (B_{\beta\gamma}^l + G_{l,k}^j B_\beta^j B_\gamma^k). \quad \dots(3.8)$$

this satisfies $H_{\beta\gamma} = H_{\gamma\beta}$, $H_{0\beta} = H_\beta$ and

$$\dot{\partial} H_\beta = H_{\beta\gamma} + H_\beta M_\gamma. \quad \dots(3.9)$$

Putting further

$$T_{\beta\gamma}^\alpha = H_0 (\dot{\partial}_\gamma M_\beta^\alpha + M_\gamma M_\beta^\alpha) + 2(H_\beta M_\gamma^\alpha + H_\gamma M_\beta^\alpha) \quad \dots(3.10)$$

the equation (3.6) is written as

$$G_{\beta\gamma}^\alpha = F_{\beta\gamma}^\alpha + T_{\beta\gamma}^\alpha, \quad \dots(3.11)$$

and the tensor $T_{\beta\gamma}^\alpha$ satisfies

$$T_{\beta\gamma}^\alpha = T_{\gamma\beta}^\alpha, \quad T_{\beta 0}^\alpha = H_0 M_\beta^\alpha, \quad T_{\beta\gamma}^0 = -H_0 M_{\beta\gamma}, \quad \dots(3.12)$$

because of (2.15) and homogeneity.

4. RELATIVE COVARIANT DIFFERENTIATIONS

The relative h - and ν -covariant derivatives of a mixed tensor Y_α^i , for example, are defined by

$$Y_{\alpha;\beta}^i = \delta_\beta Y_\alpha^i + Y_\alpha^j G_{jk}^i B_\beta^k - Y_\gamma^i G_{\alpha\beta}^\gamma, Y_{\alpha;\beta}^i = \partial_\beta Y_\alpha^i, \quad \dots(4.1)$$

where $\delta_\beta = \partial_\beta - G_\beta^\gamma \partial_\gamma$. It is obvious that the relative covariant derivatives of a tensor of the hypersurface coincide with the usual covariant derivatives with respect to the intrinsic Berwald connection. By virtue of (3.7), (3.8) and (3.11) we have

$$B_{\alpha;\beta}^i = H_{\alpha\beta} B^i - T_{\alpha\beta}^\gamma B_\gamma^i. \quad \dots(4.2)$$

On the other hand, because of $\partial_\beta B_\alpha^i = 0$, we have

$$B_{\alpha;\beta}^i = 0. \quad \dots(4.3)$$

Equations (4.2) and (4.3) are thought of as the Gauss formulas in the hypersurface theory of Finsler geometry.

Now we consider the relation between δ_β and δ_j . It follows from (2.9), (2.12), (3.3) and (3.4) that

$$\delta_\beta = B_\beta^i \delta_j + (H_\beta B^j - H_0 M_\beta^\epsilon B_\epsilon^j) \partial_j. \quad \dots(4.4)$$

We use this relation to obtain

$$Y_{j;\beta}^i = B_\beta^k Y_{j;k}^i + (H_\beta B^k - H_0 M_\beta^\epsilon B_\epsilon^k) Y_{j;k}^i, Y_{j;\beta}^i = B_\beta^k Y_{j;k}^i \quad \dots(4.5)$$

for a tensor field Y_j^i of M^n defined on M^{n-1} . Especially we have

$$g_{i;\beta} = -2C_{ijk} B_\beta^k + 2C_{ijk} (H_\beta B^k - H_0 M_\beta^\epsilon B_\epsilon^k), \quad g_{i;j;\beta} = 2C_{ijk} B_\beta^k. \quad \dots(4.6)$$

Differentiating covariantly two equations $g_{ij} B_\alpha^i B^j = 0$, $g_{ij} B^i B^j = 1$, and using (4.2), (4.3) and (4.6), we have the Weingarten formulas

$$B_{j;\beta}^i = (-H_\beta^\alpha + 2Q_\beta^\alpha - 2M^\alpha H_\beta + 2H_0 M_\epsilon^\alpha M_\beta^\epsilon) B_\alpha^j + (Q_\beta - MH_\beta + H_0 M_\epsilon M_\beta^\epsilon) B^j. \quad \dots(4.7)$$

$$B_{\beta}^j = -2M_{\beta}^{\alpha} B_{\alpha}^j - M_{\beta} B^j. \quad \dots(4.8)$$

where we put $Q_{\alpha\beta} = C_{ijk} |_0 B_{\alpha}^{ij} B^k$ and $Q_{\beta} = C_{ijkl} B_{\beta}^i B^j B^k$.

Summarizing up the above we have

Theorem 1—With respect to the intrinsic Berwald connection of a Finsler hypersurface, the Gauss formulas and the Weingarten formulas are given by (4.2), (4.3) and (4.7), (4.8), respectively.

5. GAUSS-CODAZZI-RICCI EQUATIONS

We first prepare commutation formulas of δ_{β} and ∂_{γ} as follows:

$$(\delta_{\gamma} \delta_{\beta} - \delta_{\beta} \delta_{\gamma}) Y_{\alpha}^i = -\partial_{\epsilon} Y_{\alpha}^i R_{\beta\gamma}^{\epsilon}, \quad \dots(5.1)$$

$$(\partial_{\gamma} \delta_{\beta} - \delta_{\beta} \partial_{\gamma}) Y_{\alpha}^i = -\partial_{\epsilon} Y_{\alpha}^i G_{\beta\gamma}^{\epsilon}. \quad \dots(5.2)$$

These formulas are easily proved by the definitions of $R_{\beta\gamma}^{\epsilon}$ and $G_{\beta\gamma}^{\epsilon}$. Then direct calculation leads to Ricci identities of the relative covariant derivatives.

$$\begin{aligned} Y_{\alpha;\beta;\gamma}^i - Y_{\alpha;\gamma;\beta}^i &= Y_{\alpha}^h [H_{hjk}^i B_{\beta\gamma}^{jk} + G_{hjk}^i (B_{\beta}^j B^k H_{\gamma} - H_{\beta}^j B_{\gamma}^k M_{\gamma}^{\epsilon} \\ &\quad - \beta | \gamma)] - Y_{\epsilon}^i H_{\alpha\beta\gamma}^{\epsilon} - Y_{\alpha;\epsilon}^i R_{\beta\gamma}^{\epsilon}, \end{aligned} \quad \dots(5.3)$$

$$Y_{\alpha;\beta;\gamma}^i - Y_{\alpha;\gamma;\beta}^i = Y_{\alpha}^h G_{hjk}^i B_{\beta\gamma}^{jk} - Y_{\epsilon}^i G_{\alpha\beta\gamma}^{\epsilon}, \quad \dots(5.4)$$

$$Y_{\alpha;\beta;\gamma}^i - Y_{\alpha;\gamma;\beta}^i = 0. \quad \dots(5.5)$$

Remark: We have not written the above calculation concretely, because it is done in a similar way as Matsumoto⁵. We shall only note here that δ_{β} and $G_{\beta\gamma}^{\alpha}$ differ from the induced ones and that $H_{\alpha\beta\gamma}^{\epsilon}$, $R_{\beta\gamma}^{\epsilon}$ and $G_{\alpha\beta\gamma}^{\epsilon}$ are defined analogously to (2.7), because they are intrinsic.

Now we are in a position to derive the Gauss equations and others. At first we apply the equation (5.2) to B_{α}^i . The left-hand side of (5.2) gives

$$(H_{\alpha\beta;\gamma} - T_{\alpha\beta}^{\epsilon} H_{\epsilon\gamma} + H_{\alpha\beta} A_{\gamma} - \beta | \gamma) B^i - (T_{\alpha\beta;\gamma}^{\epsilon} - T_{\alpha\beta}^{\epsilon} T_{\epsilon\gamma}^{\epsilon})$$

(equation continued on p. 852)

$$- H_{\alpha\beta} A_{\gamma}^{\epsilon}) B_{\epsilon}^i,$$

where we rewrite eqn. (4.7), for simplicity, in the form

$$B_{;\beta}^j = A_{\beta}^{\alpha} B_{\alpha}^j + A_{\beta} B^j. \quad \dots(5.6)$$

Equating the tangential or normal component of the both side, we have

$$\begin{aligned} B_{\alpha}^h B_i^{\epsilon} [H_{hjk}^i B_{\beta\gamma}^{jk} + G_{hjk}^i (B_{\beta}^j B^k H_{\gamma} - H_0 B_{\beta}^j B_{\epsilon}^k M_{\gamma}^{\epsilon} - \beta | \gamma)] \\ = H_{\alpha\beta\gamma}^{\epsilon} - (T_{\alpha\beta;\gamma}^{\epsilon} - T_{\alpha\beta}^{\epsilon} T_{\epsilon\gamma}^{\epsilon} - H_{\alpha\beta} A_{\gamma}^{\epsilon} - \beta | \gamma). \end{aligned} \quad \dots(5.7)$$

$$\begin{aligned} B_{\alpha}^h B_l [H_{hjk}^i B_{\beta\gamma}^{jk} + G_{hjk}^i (B_{\beta}^j B^k B_{\gamma} - H_0 B_{\beta}^j B_{\epsilon}^k M_{\gamma}^{\epsilon} - \beta | \gamma)] \\ = H_{\alpha\beta;\gamma} - T_{\alpha\beta}^{\epsilon} H_{\epsilon\gamma} + H_{\alpha\beta} A_{\gamma} - \beta | \gamma. \end{aligned} \quad \dots(5.8)$$

Next applying (5.2) to B^i , we get in the same way

$$\begin{aligned} B^h B_i^{\epsilon} [H_{hjk}^i B_{\beta\gamma}^{jk} + G_{hjk}^i (B_{\beta}^j B^k H_{\gamma} - H_0 B_{\beta}^j B_{\epsilon}^k M_{\gamma}^{\epsilon} - \beta | \gamma)] \\ = A_{\beta;\gamma}^{\epsilon} - A_{\beta}^{\epsilon} T_{\epsilon\gamma}^{\epsilon} + A_{\beta} A_{\gamma}^{\epsilon} - \beta | \gamma \end{aligned} \quad \dots(5.9)$$

$$\begin{aligned} B^h B_l [H_{hjk}^i B_{\beta\gamma}^{jk} + G_{hjk}^i (B_{\beta}^j B^k H_{\gamma} - H_0 B_{\beta}^j B_{\epsilon}^k M_{\gamma}^{\epsilon} - \beta | \gamma)] \\ = -M_{\epsilon} R_{\beta\gamma}^{\epsilon} + (A_{\beta;\gamma} + A_{\beta}^{\epsilon} H_{\epsilon\gamma} - \beta | \gamma). \end{aligned} \quad \dots(5.10)$$

Remark : Equations (5.7) and (5.8) may be called the Gauss and Codazzi equations. Then (5.9) and (5.10) are called the second Codazzi equation and the Ricci equation, respectively, for the curvature tensor H . The last two equations are independent of the former because the Berwald connection is not metrical, or H_{hijk} is not skew-symmetric with respect to h and i .

In the same process, from eqn. (5.3), we can get four equations for the $h\nu$ -curvature tensor C as follows:

$$B_{\alpha}^h B_i^{\epsilon} G_{hjk}^i B_{\beta\gamma}^{jk} = G_{\alpha\beta\gamma}^{\epsilon} - T_{\alpha\beta;\gamma}^{\epsilon} - 2H_{\alpha\beta} M_{\gamma}^{\epsilon} \quad \dots(5.11)$$

$$B_{\alpha}^h B_l G_{hjk}^i B_{\beta\gamma}^{jk} = H_{\alpha\beta;\gamma} - H_{\alpha\beta} M_{\gamma} \quad \dots(5.12)$$

$$\begin{aligned} B^h B_i^{\epsilon} G_{hjk}^i B_{\beta\gamma}^{jk} = A_{\beta;\gamma}^{\epsilon} - 2A_{\beta} M_{\gamma}^{\epsilon} + 2M_{\gamma;\beta}^{\epsilon} \\ - 2T_{\beta;\epsilon}^{\epsilon} M_{\gamma}^{\epsilon} + A_{\beta}^{\epsilon} M_{\gamma} \dots(5.13) \end{aligned}$$

$$B^h B_t G_{hjk}^i B_{\beta\gamma}^{ik} = A_{\beta,\gamma} + M_{\gamma;\beta} + 2H_{\beta\delta} M_{\gamma}^{\delta}. \quad \dots(5.14)$$

Summarizing up the above we have

Theorem 2—The Gauss, Codazzi, second Codazzi and Ricci equations for the intrinsic Berwald connection of a Finsler hypersurface are given by eqns. (5.7) to (5.14).

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ON THE MATRIX ROOTS OF $f(X) = A$

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For A an $n \times n$ complex valued matrix and $f(x)$ a non-constant polynomial with complex coefficients, we give conditions for the existence of an $n \times n$ complex matrix B with $f(B) = A$. The results are then applied to the study of the composition of polynomials.

Throughout this note $M_n(\mathbb{C})$ denotes the set of $n \times n$ complex valued matrices. $f(x)$ will denote a non-constant polynomial with complex coefficients. For $A \in M_n(\mathbb{C})$ we investigate the solution in $M_n(\mathbb{C})$ of the equation $f(X) = A$. The first section gives some necessary preliminary calculations, the second section outlines the solution of the problem and the last section shows an application of the results to the determination of the range of the composition of polynomials modulo an ideal of $\mathbb{C}[x]$.

§1. In this section the basic calculations are done.

Lemma 1.1—If $f(x) \in \mathbb{C}[x]$ and $B, C \in M_n(\mathbb{C})$ with C invertible, then $f(CBC^{-1}) = Cf(B)C^{-1}$.

PROOF : Straightforward.

Lemma 1.2—Let

$$B = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \\ & & & & \ddots & \\ & & 0 & & & 1 \\ & & & & & & \lambda \end{bmatrix}$$

be an $n \times n$ complex matrix and $f(x) \in \mathbb{C}[x]$. Then

$$f(B) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \frac{f'''(\lambda)}{3!} & \dots \\ & f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2!} & \dots \\ & & f(\lambda) & f'(\lambda) & \dots \\ & & & \dots & \dots & \dots \\ & O & & & f(\lambda) & f'(\lambda) \\ & & & & & f(\lambda) \end{bmatrix}.$$

PROOF : Write $f(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_0$, $a_i \in \mathbb{C}$, so that $f(B) = a_k B^k + a_{k-1} B^{k-1} + \dots + a_0$. Let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & O \\ & 0 & 1 & 0 & \dots & \\ & & 0 & 1 & \dots & \\ & O & & & & \\ & & & 0 & 1 & \\ & & & & 0 & \end{bmatrix}.$$

Then

$$N^r = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

where the 1 in the first row appears in the $(r+1)$ st position. $B = N + \lambda$ where, for simplicity, λ implies the matrix λI .

$$B^t = (N + \lambda)^t = \sum_{j=0}^t \binom{t}{j} N^j \lambda^{t-j}.$$

$$f(B) = \sum_{i=0}^k a_i B^i = \sum_{i=0}^k a_i \sum_{j=0}^i \binom{i}{j} \lambda^{i-j} N^j$$

$$= \sum_{i=0}^k \sum_{j=0}^k a_i \binom{i}{j} \lambda^{i-j} N^j \quad (\text{Since } j > i \text{ implies } \binom{i}{j} = 0)$$

(equation continued on p. 856)

$$\begin{aligned}
&= \sum_{j=0}^k \left(\sum_{i=0}^k a_i \binom{i}{j} \lambda^{i-j} \right) N^j \\
&= \sum_{j=0}^k \frac{f^{(j)}(\lambda)}{j!} N^j \text{ which is our required form.}
\end{aligned}$$

Let $A = (a_{ij})$ be an $n \times n$ complex matrix. For $k = 1, \dots, n$, we call the elements $a_{1,k}, a_{2,k+1}, \dots, a_{n-k+1,n}$ the $(k-1)$ -st upper diagonal of A . The main diagonal of A is then the 0-th upper diagonal. We will say a_{ij} is below the $(k-1)$ st upper diagonal if either $j < k$ or $j \geq k$ and $i > (1+j) - k$.

The basic lemma for the solution of our problem is the following.

Lemma 1.3—Let $1 \leq k \leq n-1$ and $B \in M_n(\mathbb{C})$. Suppose each element of the k -th upper diagonal of B is non-zero and each element below the k th upper diagonal is zero. Write $n = qk + r$ with $1 \leq r \leq k$. Then the Jordan form of B consists of k nilpotent blocks, r of them $(q+1) \times (q+1)$ and the remaining $k-r$, $q \times q$.

PROOF : Write $B = (b_{ij})$ and let B be the matrix of a transformation T from V to V (V denotes an n -dimensional complex vector space) with respect to a basis v_1, v_2, \dots, v_n of V ; i. e. $Tv_l = \sum_{i=1}^n b_{il} v_i$ for $l = 1, 2, \dots, n$. Because each element below the k th diagonal of B is zero, coupled with the assumption that each element on the k th upper diagonal is non-zero, we have that $Tv_{k+1}, Tv_{k+2}, \dots, Tv_n, v_{n-k+1}, v_{n-k+2}, \dots, v_n$ is a basis of V . The result will follow if we can find k T -invariant subspaces U_1, U_2, \dots, U_k with the properties :

$$(a) \dim(U_i) = q+1, 1 \leq i \leq r$$

$$\dim(U_i) = q, r+1 \leq i \leq k$$

$$(b) V = U_1 \oplus U_2 \oplus \dots \oplus U_k$$

(c) For each $i, 1 \leq i \leq k$, the degree of the minimum polynomial to T restricted to U_i equals the dimension of U_i .

For $1 \leq i \leq r$, set U_i to be the subspace spanned by $v_{n-i+1}, Tv_{n-i+1}, T^2v_{n-i+1}, \dots, T^{q-1}v_{n-i+1}$, and for $r+1 \leq i \leq k$, let U_i be the subspace spanned by $v_{n-i+1}, Tv_{n-i+1}, \dots, T^{q-1}v_{n-i+1}$ where we have interpreted T^{q-1} to be the identity if $q=1$.

Since Tv_l is a linear combination of v_1, v_2, \dots, v_{l-k} for $l \leq k+1$, with the coefficient of v_{l-k} non-zero, while $Tv_l = 0$ if $l \leq k$, we easily deduce by induction that T^jv_l is a linear combination of $v_1, v_2, \dots, v_{l-jk}$, with $v_l = 0$ if $l \leq 0$. So T^jv_l is zero if and only if $l-jk \leq 0$. If $n-r+1 \leq i \leq n$, T^qv_i is non-zero since $n+1 > qk+r$

while $T^{q+1} v_i = 0$ because $n + 1 \leq (q + 1)k + r$. Similarly, if $n - k + 1 \leq i \leq n - r$, $T^{q-1} v_i \neq 0$ but $T^q v_i = 0$. This implies that the subspaces $U_i, i = 1, \dots, k$, are T -invariant cyclic subspaces. But it is easy to check (using the basis $Tv_{k+1}, \dots, Tv_n, v_{n-k+1}, v_{n-k+2}, \dots, v_n$) that the collection $v_n, Tv_n, \dots, T^q v_n, v_{n-1}, Tv_{n-1}, \dots, T^q v_{n-1}, \dots, v_{n-r+1}, Tv_{n-r+1}, \dots, T^q v_{n-r+1}, v_{n-r}, Tv_{n-r}, \dots, T^{q-1} v_{n-r}, \dots, v_{n-k+1}, Tv_{n-k+1}, \dots, T^{q-1} v_{n-k+1}$ is also a basis of V . The result now follows.

§2. *Proposition 2.1*—Let $A \in M_n(\mathbb{C})$, $f(x) \in C[x]$ and $\lambda_i, i = 1, \dots, k$ be distinct complex numbers. Suppose A is similar to

$$\begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_k \end{bmatrix} \quad \text{where } A \text{ is an } n_i \times n_i \text{ matrix having } \lambda_i \text{ as its only}$$

eigenvalue. There exists a $B \in M_n(\mathbb{C})$ such that $f(B) = A$ if and only if there exists for $i = 1, \dots, k$, $B_i \in M_{n_i}(\mathbb{C})$ such that $f(B_i) = A_i$.

PROOF : If there exist B_1, B_2, \dots, B_k with $B_i \in M_{n_i}(\mathbb{C})$ such that $f(B_i) = A_i$, then

$$B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{bmatrix}$$

is an $n \times n$ complex matrix for which $f(B)$ is similar to A . By Lemma 1.1, there is then a solution to $f(X) = A$.

Conversely, suppose there is $B \in M_n(\mathbb{C})$ such that $f(B) = A$. Let J denote the Jordan form of B . By Lemma 1.1, $f(J)$ is a matrix similar to A and so, by Lemma 1.2, the eigenvalues of B are solutions of $f(x) = \lambda_i, i = 1, \dots, k$. Let D_i be the sum of all Jordan blocks of J which have eigenvalues obtained by solving $f(x) = \lambda_i$. We can then assume

$$J = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_k \end{bmatrix} \quad \text{Because}$$

$$f(J) = \begin{bmatrix} f(D_1) & & \\ & f(D_2) & \\ & & \ddots \\ & & & f(D_k) \end{bmatrix}$$

the Jordan form of it, being the sum of the Jordan forms of $f(D_i)$, $i = 1, \dots, k$, must have the sum of all blocks with λ_i on the main diagonal as the Jordan form of $f(D_i)$. Like reasoning for A_i implies that A_i and $f(D_i)$ for $i = 1, \dots, k$, are similar as they have the same Jordan form. For $i = 1, \dots, k$, let $C_i \in M_{n_i}(\mathbb{C})$ be such that $C_i f(D_i) C_i^{-1} = A_i$ and $B_i = C_i D_i C_i^{-1}$. By Lemma 1.1, the result now follows.

In view of Proposition 2.1, we may assume that A has only one eigenvalue λ , in finding solutions to $f(X) = A$.

Suppose A is an $n \times n$ complex matrix with λ its only eigenvalue. We say that A is of type (a_1, a_2, \dots, a_n) if the Jordan form of A consists of a_1 1×1 blocks, a_2 2×2 blocks, \dots , a_n $n \times n$ blocks, where we have called the $i \times i$ matrix

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \lambda & 1 & \\ & & & \ddots & \\ 0 & & & & \lambda \end{bmatrix} \quad \text{an } i \times i \text{ block. Note that } a_i \geq 0 \text{ and } \sum_{i=1}^n i a_i = n.$$

We will call the $n \times n$ matrix B an irreducible matrix if B is not similar to a matrix of the form

$$\begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

where B_1 and B_2 are square matrices.

We say $f(X) = A$ has an irreducible solution if there exists an irreducible matrix $B \in M_n(f)$ such that $f(B) = A$. By Proposition 2.1, in order for $f(X) = A$ to have an irreducible solution, A must have a single eigenvalue. Of course, if A is a 1×1 matrix, there exists an irreducible solution to $f(X) = A$.

Proposition 2.2— Let $n \geq 2$, and A an $n \times n$ complex matrix having a single eigenvalue λ . Suppose that A is of type (a_1, a_2, \dots, a_n) . Let $t = \max_{1 \leq i \leq n} \{i \mid a_i \neq 0\}$ and

write $k = a_t + a_{t-1}$. (If $t = 1$, call $a_0 = 0$.) The equation $f(X) = A$ has an irreducible solution if and only if

- (i) $n = kt - a_{t-1}$
- (ii) there exists a $\gamma \in \mathbb{C}$ such that $f(\gamma) = \lambda$, $f'(\gamma) = \dots = f^{(k-1)}(\gamma) \neq 0$, and $f^{(k)}(\gamma) \neq 0$.

PROOF : Suppose there is an irreducible solution B of $f(X) = A$. Then the Jordan form \bar{B} of B is

$$\begin{bmatrix} \gamma & 1 & & & O \\ & \gamma & 1 & & \\ & & \gamma & 1 & \\ & & & \ddots & \\ O & & & & 1 \\ & & & & \gamma \end{bmatrix}$$

for some $\gamma \in \mathbb{C}$. $f(\bar{B})$ is similar to A , and $f(\gamma) = \lambda$. Suppose γ has the property that $f^{(j)}(\gamma) \neq 0$ but $f'(\gamma) = \dots = f^{(j-1)}(\gamma) = 0$. By Lemmas 1.2 and 1.3, the Jordan form of $f(B)$ has Jordan blocks only of size $(q+1) \times (q+1)$ and $q \times q$ where q satisfies the condition $n = qj + r$ with $1 \leq r \leq j$. There are r blocks of size $(q+1) \times (q+1)$ and $j-r$ blocks of size $q \times q$. But $f(B)$ and A have the same Jordan form and so $j = k$, $q = t-1$, $r = a_t$ and the result follows.

Conversely suppose $n = kt - a_{t-1}$ and $\gamma \in \mathbb{C}$ with $f(\gamma) = \lambda$.

$$f'(\gamma) = \dots = f^{(k-1)}(\gamma) = 0, f^{(k)}(\gamma) \neq 0.$$

Let

$$B = \begin{bmatrix} \gamma & 1 & & & O \\ & \gamma & 1 & & \\ & & \gamma & 1 & \\ & & & \ddots & \\ & & & & \gamma & 1 \\ O & & & & & \gamma \end{bmatrix}$$

From Lemmas 1.2 and 1.3 we see that $f(B)$ is similar to A and from Lemma 1.1 the proposition follows.

Corollary 2.3—Let $A \in M_n(\mathbb{C})$ and suppose A is similar to

$$\begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \\ & & & & \lambda & 1 \\ 0 & & & & & \lambda \end{bmatrix} \quad \text{with } \lambda \in \mathbb{C}$$

There exists $B \in M_n(\mathbb{C})$ such that $f(B) = A$ if and only if

- (i) $n = 1$ or
- (ii) there exists $\gamma \in \mathbb{C}$, such that $f(\gamma) = \lambda$ and $f'(\gamma) \neq 0$.

PROOF : If $n = 1$, set B to be a root of $f(x) = \lambda$. Suppose $n \geq 2$. Then A is of type $(0, 0, \dots, 0, 1)$. If there exists a solution $B \in M_n(\mathbb{C})$ such that $f(B) = A$, B is irreducible as A is. By Proposition 2.2, $k = 1$ and there exists $\gamma \in \mathbb{C}$ with $f(\gamma) = \lambda$ and $f'(\gamma) \neq 0$. Conversely, if there is a $\gamma \in \mathbb{C}$ with $f(\gamma) = \lambda$ and $f'(\gamma) \neq 0$ Proposition 2.2 implies there is a solution to $f(X) = A$.

Corollary 2.4—Let $A \in M_n(\mathbb{C})$ and suppose λ is the only eigenvalue for A . If there is a $\gamma \in \mathbb{C}$ with $f(\gamma) = \lambda$ and $f'(\gamma) \neq 0$ then there exists $B \in M_n(\mathbb{C})$ such that $f(B) = A$.

PROOF : The Jordan form of A consists of a sum of blocks A_i of the type in Corollary 2.3. Each block gives a solution B_i of $f(X) = A_i$ and a matrix similar to the diagonal sum of B_i 's is a solution of $f(X) = A$.

Corollary 2.5—Let $A \in M_n(\mathbb{C})$ and suppose A is diagonalizable. Then there exists $B \in M_n(\mathbb{C})$ such that $f(X) = A$.

PROOF : Since A is similar to a diagonal sum of 1×1 matrices by Corollary 2.3 we have a solution to $f(X) = A$.

Now let A be an $n \times n$ complex matrix and $f(x)$ a non-constant complex coefficient polynomial. To see if $f(x) = A$ has a solution $B \in M_n(\mathbb{C})$ we employ the following procedure. Break up A , or a matrix similar to A , into the diagonal sum of matrices, A_1, A_2, \dots, A_k where A_i is a matrix having λ_i as its only eigenvalue and $\lambda_i \neq \lambda_j$ for $i \neq j$. By Proposition 2.1, $f(X) = A$ has a solution if and only if $f(X) = A_i$, $i = 1, \dots, k$, each have one. To solve $f(X) = A_i$, we must find, say l_i irreducible matrices $B_{i_1}, B_{i_2}, \dots, B_{i_{l_i}}$ such that $\sum_{j=1}^{l_i} \dim(B_{i_j}) = \dim A_i$ and with the additional property that

$$\left[\begin{array}{cccc} f(B_{i_1}) & & & \\ & f(B_{i_2}) & & \\ & & \ddots & \\ & & & f(B_{i_l}) \end{array} \right] \text{ is similar to } A_i.$$

Conditions for this are implied by Proposition 2.2 and Corollary 2.3. We now illustrate the technique outlined above.

Proposition 2.6—Suppose $1 \geq t$, $a, \gamma \in \mathbb{C}$ with $a \neq 0$ and A is a non-scalar matrix in $M_n(\mathbb{C})$ with λ its only eigenvalue. Let (a_1, a_2, \dots, a_n) be the type of A , and write $b_i = \sum_{j=1}^n a_j$, $i = 1, \dots, n$. Suppose further $f(x) = a(x - \gamma)^r + \lambda$. Then $f(X) = A$ has a solution in $M_n(\mathbb{C})$ if and only if for each i

$$b_i \equiv 0 \pmod{r} \text{ or } i > 1 \text{ and } \left\lfloor \frac{b_{i-1}}{r} \right\rfloor > \left\lfloor \frac{b_i}{r} \right\rfloor.$$

($\lfloor x \rfloor$ denotes the greatest integer in x).

PROOF : If $n = 2$, then the theorem follows by Proposition 2.2. We can thus assume, via induction, that the theorem is true for smaller values of n . Assume there is a solution B of $f(X) = A$. If B is irreducible the result will follow from Proposition 2.2. Otherwise, there exist square matrices A_1 and A_2 such that A is similar to

$$\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}$$

and square matrices B_1 and B_2 with B similar to

$$\begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

and $f(B_k) = A_k$, $k = 1, 2$. Suppose the type of A_k , $k = 1, 2$, is $(a_{k_1}, a_{k_2}, \dots, a_{k_n})$ and

$b_k = \sum_{j=1}^n a_{k_j}$. Then

$$a_i = a_{1_i} + a_{2_i} \text{ and } b_i = b_{1_i} + b_{2_i}$$

for $i = 1, \dots, n$. By induction for $k = 1, 2$, $b_{k_i} \equiv 0 \pmod{r}$ or

$i > 1$ and $\left\lfloor \frac{b_{k_{i-1}}}{r} \right\rfloor > \left\lfloor \frac{b_{k_i}}{r} \right\rfloor$ and it is easy to check that the conditions of the theorem are valid for the sequence b_i .

Conversely, if the conditions of the proposition on the b_i sequence hold and $t = \{\max i \mid a_i \neq 0\}$ then either $a_t \geq r$ or $a_t < r$ and $a_t + a_{t-1} \geq r$. In the first case let A_1 be an $(n - rt) \times (n - rt)$ matrix with type $(a_1, a_2, \dots, a_{t-1}, a_t - r, a_{t+1}, \dots, a_n)$ and A_2 an $rt \times rt$ matrix with type $(0, 0, \dots, 0, r, 0, \dots, 0)$ where the entry r is in the t th position, while in the second case let A_1 be a matrix with type

$(a_1, a_2, \dots, a_{t-2}, a_{t-1}, +a_t - r, 0, a_{t+1}, \dots, a_n)$ and A_2 a matrix with type $(0, 0, \dots, 0, r - a_t, a_t, 0, \dots, 0)$. In either case A is similar to

$$\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}.$$

If $A_2 = A$, then A has an irreducible solution by Proposition 2.2. Otherwise the b sequences for A_1 and A_2 satisfy the hypothesis of the proposition and so there exists B_1 and B_2 square matrices such that $f(B_k) = A_k$, $k = 1, 2$. This implies that a matrix similar to

$$\begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

gives the required result.

Corollary 2.7—Let $A \in M_n(\mathbb{C})$. A is diagonalizable if and only if each non-constant polynomial $f(x) \in \mathbb{C}[x]$ has a solution $B = B_f \in M_n(\mathbb{C})$ of $f(X) = A$.

PROOF: If A is diagonalizable and $f(x) \in \mathbb{C}[x]$ with $f(x)$ of degree at least one, by Corollary 2.5, the equation $f(x) = A$ has a solution. If A is not diagonalizable, for some $\lambda \in \mathbb{C}$, $(x - \lambda)^2$ divides the minimal polynomial for A . By Proposition 2.6, if $\gamma \in \mathbb{C}$, the equation $(x - \gamma)^{n+1} + \lambda = A$ has no solution $B \in M_n(\mathbb{C})$.

Corollary 2.8—Suppose $f(x) \in \mathbb{C}[x]$ is a polynomial of degree 2 and A is an $n \times n$ matrix with λ its only eigenvalue and type (a_1, a_2, \dots, a_n) . Let $b_i = \sum_{j=1}^n a_j$ for $i = 1, \dots, n$. Then there is an $n \times n$ matrix B such that $f(B) = A$ if and only if either

(i) $f(x) - \lambda = 0$ has distinct roots or

(ii) $f(x) - \lambda = 0$ has a double root and for $i = 1, \dots, n$, $b_i \equiv 0 \pmod{2}$ or

$$i > 1 \text{ and } \left\lfloor \frac{b_{i-1}}{2} \right\rfloor > \left\lfloor \frac{b_i}{2} \right\rfloor.$$

PROOF: If $f(x) = A$ has a solution and $f(x) - \lambda = 0$ does not have distinct roots, then $f(x) = a(x - \gamma)^2 + \lambda$ for $a, \gamma \in \mathbb{C}$ with $a \neq 0$. By Proposition 2.6 $b_i \equiv 0 \pmod{2}$ or $i > 1$ and $\left\lfloor \frac{b_{i-1}}{2} \right\rfloor > \left\lfloor \frac{b_i}{2} \right\rfloor$ for $i = 1, \dots, n$.

Conversely, if $f(x) - \lambda = 0$ has distinct roots then $f(X) = A$ has a solution by Corollary 2.4 while if condition (ii) is met, then there is a solution to $f(X) = A$ via Proposition 2.6.

Corollary 2.9—Suppose $f(x) \in \mathbb{C}[x]$ is a polynomial of degree 3 and A is an $n \times n$ matrix with λ its only eigenvalue and type (a_1, a_2, \dots, a_n) . Let $b_i = \sum_{j=1}^n a_j$ for $i = 1, \dots, n$. Then there is an $n \times n$ matrix B such that $f(B) = A$ if and only if

- (i) $f(x) - \lambda = 0$ does not have a triple root, or
- (ii) $f(x) - \lambda = 0$ has a triple root and for $i = 1, \dots, n$,

$$b_i \equiv 0 \pmod{3} \text{ or } i > 1 \text{ and } \left\lfloor \frac{b_{i-1}}{3} \right\rfloor > \left\lfloor \frac{b_i}{3} \right\rfloor.$$

PROOF: If $f(x) = A$ has a solution and $f(x) - \lambda = 0$ does have a triple root, the Proposition 2.6 implies condition (ii) holds. Conversely, if $f(x) - \lambda = 0$ does not have a triple root, then there must be a root of $f(x) - \lambda = 0$ such that $f'(\gamma) \neq 0$. By Corollary 2.4, $f(X) = A$ then has a solution. If condition (ii) is met, then there is a solution to $f(X) = A$ by Proposition 2.6.

§3. As an illustration of the preceding results, we investigate in this section the composition of two polynomials modulo a given ideal.

Let I denote a non-zero ideal in $\mathbb{C}[x]$ which is generated by $m(x)$ a monic non-constant polynomial. Suppose $f(x), h(x) \in \mathbb{C}[x]$ with $f(x)$ of degree at least one. Does there exist $g(x) \in \mathbb{C}[x]$ such that $f(g(x)) \equiv h(x) \pmod{I}$? Two contrasting possibilities for $h(x)$ are considered, namely $h(x) = 0$ and $h(x) = x$.

Proposition 3.1—If $f(x)$ is a non-constant polynomial in $\mathbb{C}[x]$ and I is a proper ideal in $\mathbb{C}[x]$, then there exists $g(x) \in \mathbb{C}[x]$ such that $f(g(x)) \equiv 0 \pmod{I}$.

PROOF: As $\mathbb{C}[x]$ is a principal ideal domain we can suppose I is generated by a monic polynomial $m(x)$. Let c be a root of $f(x) = 0$. If $s(x) \in \mathbb{C}[x]$, then $g(x) = s(x)m(x) + c$ satisfies the proposition.

Suppose now $m(x) = (x - \lambda_1)^{e_1} (x - \lambda_2)^{e_2} \dots (x - \lambda_k)^{e_k}$ with $e_i > 0$ for $i = 1, \dots, k$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Let $n = e_1 + e_2 + \dots + e_k$. Finally let A be an $n \times n$ matrix with minimal polynomial $m(x)$. In terms of this notation we show

Lemma 3.2—There exists $g(x) \in \mathbb{C}[x]$ such that $f(g(x)) \equiv x \pmod{I}$ if and only if there exists $B \in M_n(\mathbb{C})$ such that $f(B) = A$.

PROOF: Suppose there is a $B \in M_n(\mathbb{C})$ such that $f(B) = A$. Then A , being a polynomial in B , commutes with B . As the characteristic polynomial of A agrees with its minimal polynomial, A is non-derogatory and so there exists a $g(x) \in \mathbb{C}[x]$ such

that $B = g(A)$ (see Marcus², p. 9), Theorem 2.14). $f(g(A)) - A = 0$ implies that $m(x)$ divides $f(g(x)) - x$ or $f(g(x)) \equiv x \pmod{I}$.

Conversely, if $f(g(x)) \equiv x \pmod{I}$ then $m(x)$ divides $(f(g(x)) - x)$ and $f(g(A)) - A = 0$. Let $B = g(A)$.

Proposition 3.3—Let $f(x)$ be a non-constant polynomial in $\mathbb{C}[x]$, and $m(x) = (x - \lambda_1)^{e_1} \dots (x - \lambda_k)^{e_k}$ with $e_i > 0$ for $i = 1, \dots, k$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Also let I be the ideal generated by $m(x)$. Then there exists $g(x) \in \mathbb{C}[x]$ such that $f(g(x)) \equiv x \pmod{I}$ if and only if for each $i = 1, \dots, k$ either $e_i = 1$ or there exists $\gamma_i \in \mathbb{C}$ such that $f(\gamma_i) = \lambda_i$ and $f'(\gamma_i) \neq 0$.

PROOF : A is similar to

$$\begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ & & & A_R \\ O & & & & A_R \end{bmatrix}$$

where

$$A_i = \begin{bmatrix} \lambda_i & 1 & & O \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ O & & & \lambda_i \end{bmatrix}$$

is an $e_i \times e_i$ matrix,

since A is non-derogatory. Lemma 3.2 together with Corollary 2.3 implies the result.

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CONVEX HULLS AND EXTREME POINTS OF SOME FAMILIES OF MULTIVALENT FUNCTIONS

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In this paper, the p -valent families $S(p, \alpha, z_0)$, $S_R(p, \alpha, z_0)$ and $S[p, \alpha, z_0]$, of starlike functions of order α , are considered. Functions in all the above families satisfy $f(0) = 0$ and $f(z_0) = z_0^p$ for a fixed z_0 , $|z_0| < 1$. In addition, functions in $S_R(p, \alpha, z_0)$ take real values on $(-1, 1)$ and all the Taylor coefficients, starting from $(p+1)$ st on, of functions in $S[p, \alpha, z_0]$, are negative. Closed convex hulls and their extreme points for these families have been obtained.

1. INTRODUCTION

In the present paper we are concerned with the determination of closed convex hulls and extreme points of families of multivalent functions. A function f analytic in the unit disc $E = \{z: |z| < 1\}$ and given by the power series

$$f(z) = a_0 z^p + \sum_{n=1}^{\infty} a_n z^{n+p}, \quad p = 1, 2, \dots, |z| < 1, a_0 \neq 0 \quad \dots(1.1)$$

is said to be in the class $S(p, \alpha, z_0)$, $\alpha < 1$, $|z_0| < 1$ if it satisfies

$$f(z_0) = z_0^p \quad \dots(1.2)$$

and

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > p\alpha, \quad |z| < 1. \quad \dots(1.3)$$

It is easy to verify that a function f is in $S(p, \alpha, z_0)$ if and only if there exists a univalent starlike function g of order α having fixed points at $z = 0$ and at $z = z_0$ such that $f(z) = (g(z))^p$. Hence functions in $S(p, \alpha, z_0)$ are p -valent starlike of order α

(Kapoor and Mishra²). Let $K(p, \alpha, z_0)$ denote the class of functions given by (1.1) and such that $zf'(z)/p$ is in $S(p, \alpha, z_0)$. It now follows that functions in $K(p, \alpha, z_0)$ are p -valent convex of order α (Kapoor and Mishra²). Further for $-1 < z_0 < 1$ and for any family F of functions analytic in E , let F_R denote the class of functions in F that take real values on the interval $(-1, 1)$. The classes $S_R(p, \alpha, z_0)$ and $K_R(p, \alpha, z_0)$ are now defined accordingly.

Fait and Zlotkiewicz¹ have determined the closed convex hulls and their extreme points of the univalent families $S(1, \alpha, z_0)$ and $K(1, \alpha, z_0)$. They have shown that the closed convex hulls can be generated by integrals of some familiar functions and the kernels in these integrals are precisely the extreme points. In Section 2 we generalize their results for the corresponding p -valent families. Specifically we determine the closed convex hulls and their extreme points for the families $S(p, \alpha, z_0)$, $K(p, \alpha, z_0)$, $S_R(p, \alpha, z_0)$ and $K_R(p, \alpha, z_0)$. This serves as a continuation of the study carried out earlier (Kapoor and Mishra)².

We next show that the above families of functions behave particularly nicely if we consider only such functions whose $(p+1)^{\text{th}}$ coefficients on, are all negative. For $-1 < z_0 < 1$, let $S[p, \alpha, z_0]$ denote the subclass of functions f in $S_R(p, \alpha, z_0)$ given by the power series

$$f(z) = a_0 z^p - \sum_{n=1}^{\infty} a_n z^{n+p}, \quad a_n \geq 0. \quad \dots(1.4)$$

Also let $K[p, \alpha, z_0]$, $-1 < z_0 < 1$, denote the subclass of functions f in $K_R(p, \alpha, z_0)$ given by (1.4). Silverman^{6,7} studied the behaviour of the univalent families $S[1, \alpha, 0]$, $K[1, \alpha, 0]$, $S[1, \alpha, z_0]$ and $K[1, \alpha, z_0]$. His results generalized the corresponding results for univalent polynomials by Schild⁵ and Pilat⁴. In Section 3 of our present paper we extend the results of Silverman to p -valent cases.

It turns out that the extreme points of the families $S[p, \alpha, z_0]$ and $K[p, \alpha, z_0]$ are much simpler in form than those of $S_R(p, \alpha, z_0)$ and $K_R(p, \alpha, z_0)$.

Throughout in this paper for any family F , we denote by HF the closed convex hull of F and by EHF the set of extreme points of HF .

2. DETERMINATION OF THE CLOSED CONVEX HULLS AND THEIR EXTREME POINTS FOR THE FAMILIES

$S(p, \alpha, z_0)$, $K(p, \alpha, z_0)$, $S_R(p, \alpha, z_0)$ AND $K_R(p, \alpha, z_0)$

Closely related to $S(p, \alpha, z_0)$ is the class of normalised p -valent starlike functions of order α . A function f analytic in E and given by the powers series $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p}$, is said to be in $S(p, \alpha)$ if it satisfies (1.3)². The following lemma establishes a one-to-one relation between $S(p, \alpha)$ and $S(p, \alpha, z_0)$.

Lemma — If f is in $S(p, \alpha)$ then g defined by

$$g(z) = \frac{z^p (1 - |z_0|^2)^{2(1-\alpha)p}}{(z - z_0)^p (1 - \bar{z}_0 z)^{p(1-2\alpha)}} f\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right), |z| < 1, \quad \dots(2.1)$$

is in $S(p, \alpha, z_0)$ and vice-versa.

PROOF : For ρ real, $0 < \rho < 1$, let

$$g_\rho(z) = \frac{z^p (1 - |z_0|^2)^{2p(1-\alpha)}}{(z - z_0)^\rho (1 - \bar{z}_0 z)^{p(1-2\alpha)}} f\left(\rho \left(\frac{z - z_0}{1 - \bar{z}_0 z}\right)\right), |z| < 1.$$

Then

$$\begin{aligned} \frac{zg'_\rho(z)}{g_\rho(z)} &= \frac{-pz_0(1 - \bar{z}_0 z) + p(1 - 2\alpha)\bar{z}_0 z(z - z_0)}{(z - z_0)(1 - \bar{z}_0 z)} \\ &\quad + \frac{z(1 - |z_0|^2)}{(z - z_0)(1 - \bar{z}_0 z)} \frac{w(z)f'(w(z))}{f(w(z))}, w(z) = \frac{\rho(z - z_0)}{(1 - \bar{z}_0 z)} \end{aligned}$$

Letting $z = e^{i\theta}$ we have

$$\begin{aligned} \frac{zg'_\rho(z)}{g_\rho(z)} &= \frac{p\{z(\bar{z}_0 z - z_0 \bar{z}) + 2\alpha z(|z_0|^2 - \bar{z}_0 z)\}}{(z - z_0)(1 - \bar{z}_0)} \\ &\quad + \frac{z(1 - |z_0|^2)}{(z - z_0)(1 - \bar{z}_0 z)} \frac{w(z)f'(w(z))}{f(w(z))} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} \frac{zg'_\rho(z)}{g_\rho(z)} &= \frac{2p\alpha(|z_0|^2 - \operatorname{Re}(\bar{z}_0 z))}{|z - z_0|^2} + \frac{1 - |z_0|^2}{|z - z_0|^2} \\ &\quad \times \operatorname{Re} \frac{w(z)f'(w(z))}{f(w(z))} \\ &> \frac{2p\alpha(|z_0|^2 - \operatorname{Re}(\bar{z}_0 z))}{|z - z_0|^2} + \frac{1 - |z_0|^2}{|z - z_0|^2} p^\alpha \\ &= \frac{p\alpha(1 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z))}{|z - z_0|^2} \\ &= p^\alpha. \end{aligned}$$

Thus $g_\rho(z)$ is a p -valent starlike function of order α for each admissible value of ρ . Since the class of p -valent starlike functions is compact $g(z) = \lim_{\rho \rightarrow 1} g_\rho(z)$ is also p -

valent starlike of order α . It is easy to verify that $g(z_0) = z_0^p$.

Conversely, we can repeat the above consideration starting with a function g in $S(p, \alpha, z_0)$ defined by (2.1) to end up with the conclusion that

$$f(z) = \frac{z^p}{(z + z_0)^\rho (1 + \bar{z}_0 z)^{p(1-2\alpha)}} g\left(\frac{z + z_0}{1 + \bar{z}_0 z}\right), |z| < 1.$$

is in $S(p, \alpha)$. This completes the proof of Lemma 1.

We next prove the following theorems :

Theorem 1—Let $X = \{x : |x| = 1\}$, P be the set of probability measures on X and let F denote the class of functions f_v given by

$$f_v(z) = \int_X \frac{z^p (1 - xz_0)^{2p(1-\alpha)}}{(1 - xz)^{2p(1-\alpha)}} d\nu(x), \nu \text{ in } P, |z| < 1.$$

Then, $F = HS(p, \alpha, z_0)$ and the functions

$$z = \int \frac{z^p (1 - xz_0)^{2p(1-\alpha)}}{(1 - xz)^{2p(1-\alpha)}} = k(z, x, p, \alpha, z_0), |x| = 1.$$

are the only extreme points of $HS(p, \alpha, z_0)$.

PROOF : Each function $k(z, x, p, \alpha, z_0)$ is in $S(p, \alpha, z_0)$. Hence $\int_X k(z, x, p, \alpha, z_0) d\nu(x)$ for ν in P is in $HS(p, \alpha, z_0)$. This gives $F \subset HS(p, \alpha, z_0)$.

To prove the opposite inclusion relation it is sufficient to show that $S(p, \alpha, z_0) \subset F$, for, F is a closed convex set. Suppose that f is in $S(p, \alpha, z_0)$. Then by Lemma 1 there exists a g in $S(p, \alpha)$ such that

$$f(z) = \frac{z^p (1 - |z_0|^2)^{2p(1-\alpha)}}{(1 - \bar{z}_0 z)^{2p(1-\alpha)} (z - z_0)^p} g\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right). \quad \dots(2.2)$$

Functions in the closed convex hull of $S(p, \alpha)$ are given by the integral representation (Kapoor and Mishra², Theorem 1),

$$\int_X \frac{z^p}{(1 - xz)^{2p(1-\alpha)}} d\nu(p), \nu \text{ in } P \text{ and } |z| < 1.$$

Therefore, we can write

$$g\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right) = \int_X \frac{\left(\frac{z - z_0}{1 - \bar{z}_0 z}\right)^p}{\left(1 - x \frac{z - z_0}{1 - \bar{z}_0 z}\right)^{2p(1-\alpha)}} d\nu(x)$$

for some ν in P . Substituting the above in (2.2) we have

$$f(z) = \int_X \frac{z^p (1 - |z_0|^2)^{2p(1-\alpha)}}{(1 - \bar{z}_0 z - x(z - z_0))^{2p(1-\alpha)}} d\nu(x).$$

The transformation $x = (y - \bar{z}_0)/(1 - yz_0)$ defines a one-to-one mapping of X onto itself. Now if we define the probability measure η by

$$d\eta(y) = d\nu\left(\frac{y - \bar{z}_0}{1 - yz_0}\right) = d\nu(x) \quad \dots(2.3)$$

then

$$f(z) = \int_X \frac{z^p (1 - yz_0)^{2p(1-\alpha)}}{(1 - yz)^{2p(1-\alpha)}} d\eta(y).$$

Hence f is in F and $HS(p, \alpha, z_0) = F$.

Since the extreme points of P are the unit point mass measures, the functions $k(z, x, p\alpha, z_0)$, $|x| = 1$ are the extreme points of $HS(p, \alpha, z_0)$ and since the mapping $\nu = f_\nu$ is one-to-one, these are the only extreme points.

Theorem 2—Let X and P be as in Theorem 1 and let F be the class of functions f_ν defined on E by

$$f_\nu(z) = \int_X \left[\int_0^z \frac{pt^{p-1} (1 - x_0 t)^{2p(1-\alpha)}}{(1 - xt)^{2p(1-\alpha)}} dt \right] d\nu(x), \nu \in P.$$

Then $F = HK(p, \alpha, z_0)$ and the functions

$$z \rightarrow \int_0^z \frac{pt^{p-1} (1 - xz_0 t)^{2p(1-\alpha)}}{(1 - xt)^{2p(1-\alpha)}} dt, x \in X$$

are the only extreme points of $HK(p, \alpha, z_0)$.

PROOF : Let A_0 be the class of functions f analytic in E such that $f(0) = 0$.

The map $(Lf)(z) = p \int_0^z \frac{f(t)}{t} dt$ is a linear homeomorphism from A_0 to A_0 and

$L(S(p, \alpha, z_0)) = K(p, \alpha, z_0)$. Since linear homeomorphism preserves closed convex hull as well its extreme points the conclusion of the theorem follows from Theorem 1.

Theorem 3—Let $X = \{x : |x| = 1, \operatorname{Im} x \geq 0\}$, P be the set of probability measures on X , z_0 real, $-1 < z_0 < 1$, and let F be the class of functions f_ν defined on E by

$$f_\nu(z) = \int_X \frac{z^p (1 - xz_0)^{p(1-\alpha)} (1 - \bar{x}z_0)^{p(1-\alpha)}}{(1 - xz)^{p(1-\alpha)} (1 - \bar{x}z)^{p(1-\alpha)}} d\nu(x), \nu \in P. \quad \dots(2.4)$$

Then, $F = HS_R(p, \alpha, z_0)$ and the functions

$$z \rightarrow \frac{z^p (1 - xz_0)^{p(1-\alpha)} (1 - \bar{x}z_0)^{p(1-\alpha)}}{(1 - xz)^{p(1-\alpha)} (1 - \bar{x}z)^{p(1-\alpha)}}, x \in X$$

are precisely the extreme points of $HS_R(p, \alpha, z_0)$.

PROOF : The lines of proof of this theorem is similar to that in Theorem 1. Each function in the kernel of the integral in (2.4) is in $S_R(p, \alpha, z_0)$.

Hence

$$F \subset HS_R(p, \alpha, z_0).$$

We, next, observe that if $-1 < z_0 < 1$ the transformation (2.1) given in Lemma 1 provides a one-to-one onto correspondence between $S_R(p, \alpha)$ and $S_R(p, \alpha, z_0)$. Now, let f be in $S_R(p, \alpha, z_0)$. Then $f(z)$ can be written as

$$f(z) = \frac{z^p (1 - z_0^2)^{2p(1-\alpha)}}{(1 - z_0 z)^{p(1-\alpha)} (z - z_0)^p} g\left(\frac{z - z_0}{1 - z_0 z}\right), |z| < 1$$

for some g in $S_R(p, \alpha)$. Functions in the closed convex hull of $S_R(p, \alpha)$ are given by the integral representation (Kapoor and Mishra², Theorem 3).

$$\int_X \frac{z^p}{(1 - xz)^p (1 - \bar{x}z)^p} dv(x), v \text{ in } P.$$

Using this formula we can write

$$f(z) = \int_X \frac{z^p (1 - z_0^2)^{2p(1-\alpha)}}{(1 - z_0 z - x(z - z_0))^{p(1-\alpha)} (1 - z_0 z - \bar{x}(z - z_0))^{p(1-\alpha)}} dv(x)$$

for some v in P . For real z_0 , $-1 < z_0 < 1$, the formula $x = (y - z_0)/(1 - yz_0)$ defines a one-to-one mapping of X onto itself and if we define the probability measure η as in (2.3) then

$$f(z) = \int_X \frac{z^p (1 - yz_0)^{p(1-\alpha)} (1 - \bar{y}z_0)^{p(1-\alpha)}}{(1 - yz)^{p(1-\alpha)} (1 - \bar{y}z)^{p(1-\alpha)}} d\eta(y).$$

Hence f is in F and $HS(p, \alpha, z_0) = F$.

The uniqueness of the extreme points follows from the fact that the transformation $v \rightarrow f_v$ is one-to-one. The proof is complete.

Theorem 5—Let X , P and z_0 be defined as in Theorem 3 and let F be the class of functions f_v defined on E by

$$f_v(z) = \int_X \left[\int_0^1 \frac{pt^{p-1} (1 - xz_0)^{p(1-\alpha)} (1 - \bar{x}z_0)^{p(1-\alpha)}}{(1 - xt)^{p(1-\alpha)} (1 - \bar{x}t)^{p(1-\alpha)}} dt \right] dv, v \text{ in } P.$$

Then $F = HK_R(p, \alpha, z_0)$ and the functions

$$z \rightarrow \int \frac{pt^{p-1} (1 - xz_0)^{p(1-\alpha)} (1 - \bar{x}z_0)^{p(1-\alpha)}}{(1 - xt)^{p(1-\alpha)} (1 - \bar{x}t)^{p(1-\alpha)}} dt, x \text{ in } X$$

are precisely the extreme points of $HK_R(p, \alpha, z_0)$.

PROOF : Since $L(S_R(p, \alpha, z_0)) = K_R(p, \alpha, z_0)$ where L is the linear homomorphism

$$(Lf)(z) = \int_0^z \frac{pf(t)}{t} dt, \text{ the theorem follows from Theorem 3.}$$

3. CHARACTERISATION THEOREMS AND EXTREME POINTS

FOR THE FAMILIES $S[p, \alpha, z_0]$ AND $K[p, \alpha, z_0]$

The following characterisation theorems for $S[p, \alpha, 0]$ and $K[p, \alpha, 0]$ are needed to derive characterisation theorems for $S[p, \alpha, z_0]$ and $K[p, \alpha, z_0]$.

*Theorem A*³—A function f given by the series (1.4) is in $S[p, \alpha, 0]$ if and only if

$$\sum_{n=1}^{\infty} (n + p - p\alpha) a_n \leq a_0 p (1 - \alpha). \quad \dots(3.1)$$

*Theorem B*³—A function f given by the series (1.4) is in $K[p, \alpha, 0]$ if and only if

$$\sum_{n=1}^{\infty} \frac{(n + p)}{p} (n + p - p\alpha) a_n \leq a_0 p (1 - \alpha). \quad \dots(3.2)$$

We have the following theorems.

Theorem 5—Suppose $a_n \geq 0$ for every n . Then $f(z) = a_0 z^p - \sum_{n=1}^{\infty} a_n z^{n+p}$ is in $S[p, \alpha, z_0]$ if and only if

$$\sum_{n=1}^{\infty} a_n \left\{ \frac{n + p - p\alpha}{p(1 - \alpha)} - z_0^n \right\} \leq 1. \quad \dots(3.3)$$

PROOF : Condition (1.2) gives $a_0 = 1 + \sum_{n=1}^{\infty} a_n z_0^n$. Substituting this value of a_0 in (3.1) we get (3.3).

Corollary 1—Suppose $a_n \geq 0$ for every n . Then $f(z) = a_0 z^p - \sum_{n=1}^{\infty} a_n z^{n+p}$ is in $K[p, \alpha, z_0]$ if and only if

$$\sum_{n=1}^{\infty} a_n \left\{ \frac{(n + p)(n + p - p\alpha)}{p p (1 - \alpha)} - z_0^n \right\} \leq 1. \quad \dots(3.4)$$

PROOF : As in the theorem we have $a_0 = 1 + \sum_{n=1}^{\infty} a_n z_0^n$. Now substituting the value of a_0 in (3.2) we have (3.4).

Corollary 2— If $f(z) = a_0 z^p - \sum_{n=1}^{\infty} a_n z^{n+p}$ is in $S[p, \alpha, z_0]$ then

$$a_n \leq \frac{p(1-\alpha)}{n+p(1-\alpha)-p(1-\alpha)z_0^n}, \quad n = 1, 2, 3, \dots \quad \dots(3.5)$$

with equality for

$$f_n(z) = \frac{(n+p-p\alpha)z^p - p(1-\alpha)z^{n+p}}{(n+p-p\alpha)-p(1-\alpha)z_0^n}, \quad n = 1, 2, 3, \dots \quad \dots(3.6)$$

Unlike $S_R(p, \alpha, z_0)$ the family $S[p, \alpha, z_0]$ is convex. For, if f_1 and f_2 are in $S[p, \alpha, z_0]$ then $\lambda f_1 + (1-\lambda)f_2$, $0 < \lambda < 1$, also satisfies the coefficient inequality (3.3). Similarly, $K[p, \alpha, z_0]$ is also a convex family. In the next two theorems we determine the extreme points of these two convex families.

Theorem 6—Set

$$f_0(z) = z^p$$

and

$$f_n(z) = \frac{(n+p(1-\alpha))z^p - (p(1-\alpha))z^{p+n}}{(n+p(1-\alpha)) - (p(1-\alpha))z_0^n}, \quad n = 1, 2, 3, \dots \quad \dots(3.7)$$

Then a function $f(z)$ is in $S[p, \alpha, z_0]$ if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$, $\sum_{n=0}^{\infty} \lambda_n = 1$. Hence the extreme points of $S[p, \alpha, z_0]$ are the functions $f_n(z)$, $n = 0, 1, 2, 3, \dots$

PROOF : Suppose $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$, $\sum_{n=0}^{\infty} \lambda_n = 1$.

$$\begin{aligned} f(z) &= \lambda_0 z^p + \sum_{n=1}^{\infty} \lambda_n \frac{(n+p-p\alpha)z^p}{(n+p-p\alpha)-p(1-\alpha)z_0^n} \\ &\quad - \sum_{n=1}^{\infty} \lambda_n \frac{p(1-\alpha)z^{n+p}}{(n+p-p\alpha)-p(1-\alpha)z_0^n} \\ &= \left[\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \left(\frac{n+p(1-\alpha)}{n+p(1-\alpha)-p(1-\alpha)z_0^n} \right) \right] \end{aligned}$$

(equation continued on p. 873)

$$z^p - \sum_{n=1}^{\infty} \lambda_n \frac{p(1-\alpha)z^{n+p}}{(n+p-p\alpha)-p(1-\alpha)z_0^n}$$

Thus, the coefficients of $f(z)$ satisfy the inequality (3.3) and f is in $S[p, \alpha, z_0]$.

Conversely, suppose that $f(z) = a_0 z^p - \sum_{n=1}^{\infty} a_n z^{n+p}$ is in $S[p, \alpha, z_0]$. Now set

$$\lambda_n = \frac{n+p(1-\alpha)-(p(1-\alpha))z_0^n}{p(1-\alpha)} a_n, \quad n = 1, 2, \dots$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.$$

Using the coefficient inequality (3.5) it is seen that $0 \leq \lambda_n \leq 1$ and by (3.3) it follows that $0 \leq \lambda_0 \leq 1$. With this choice of λ_n , $f(z)$ can be written as $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$. This shows that the extreme points of $S[p, \alpha, z_0]$ are f_n for $n = 0, 1, 2, \dots$ and the proof is complete.

Theorem 7—Set

$$f_0(z) = z^p$$

and

$$f_n(z) = \frac{(n+p)(n+p(1-\alpha))z^p - p^2(1-\alpha)z^{p+n}}{(n+p)(n+p(1+\alpha)) - p^2(1-\alpha)z_0^n},$$

$$n = 1, 2, 3, \dots \quad \dots(3.8)$$

Then $f(z)$ is in $K[p, \alpha, z_0]$ if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. The extreme points of $K[p, \alpha, z_0]$ are precisely the functions f_n , $n = 0, 1, 2, \dots$.

PROOF : The lines of proof of Theorem 6 can be adopted to prove this theorem and we omit the details.

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ON LINEAR COMBINATIONS OF n ANALYTIC FUNCTIONS IN GENERALIZED PINCHUK AND GENERALIZED MOULIS CLASSES

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In this paper, we obtain two Theorems of general nature which unify and generalize some known radii results concerning linear combinations of analytic functions in various well known classes.

1. INTRODUCTION

Let N denote the set of all analytic functions on the unit disc $|z| < 1$ such that $f(0) = 0$, $f'(0) = 1$. Let $U_k^\lambda(\beta, c)$ be the class of all functions f in N such that

$$\int_0^{2\pi} |\operatorname{Re} e^{i\lambda} J_{f(z)} - \beta \cos \lambda| d\theta \leq k\pi (1 - \beta) \cos \lambda \quad \dots(1)$$

where

$$J_{f(z)} = 1 - \frac{1}{c} + \frac{z}{c} \frac{f'(z)}{f(z)}, \quad \dots(2)$$

c being a nonzero complex number, $-\pi/2 < \lambda < \pi/2$, $0 \leq \beta < 1$ and $k \geq 2$. The class $U_k^\lambda(\beta, c)$ contains as special cases many classes of analytic functions studied in literature, for example, (i) ($c = 1$, $\lambda = 0$, $k = 2$) the class $S^*(\beta)$ of starlike functions of order β due to Robertson¹³, (ii) ($c = 1$, $k = 2$) the class $U_2^\lambda(\beta)$ of λ -spirallike functions of order β due to Sizuk¹⁶, (iii) ($c = 1$, $\beta = 0$, $\lambda = 0$) the class U_k of bounded radius rotation due to Pinchuk¹² and its generalizations $U_k(\beta)$ due to Padmanabhan and Parvatham¹¹ and $U_k^\lambda(\beta)$ due to Reddy⁵.

Let $V_k^\lambda(\beta, c)$ be the class of functions f in N such that zf' is in $U_k^\lambda(\beta, c)$. This class generalizes many classes of analytic functions such as (i) ($c = 1$, $\lambda = 0$, $k = 2$) the class $C(\beta)$ of convex functions of order β due to Robertson¹³, (ii) ($c = 1$, $\beta = 0$, $k = 2$) the class V_2^λ of Robertson functions namely functions f for which zf' is λ -spirallike¹⁴, (iii) ($c = 1$, $\lambda = 0$, $\beta = 0$) the class V_k of functions of bounded boundary

rotation due to Paatero⁹ and its generalizations $V_k^\lambda(\beta)$ due to Moulis⁷ and $V_k(c)$ due to Nasr⁸. The class $V_k^\lambda(\beta, c)$ was introduced by Reddy⁵.

In Section 3 of this note we prove the following two Theorems concerning linear combinations of n functions in the classes $U_k^\lambda(\beta, c)$ and $V_k^\lambda(\beta, c)$.

Theorem 1—Let f_1, \dots, f_n be n functions in $U_k^\lambda(\beta, c)$ and $F = \gamma_1 f_1 + \dots + \gamma_n f_n$ where $\gamma_1, \dots, \gamma_n$ are complex numbers such that $0 \leq \mu = \max_{1 \leq i, j \leq n} \arg(\gamma_i/\gamma_j) < \pi$. Further let $\phi = \arg c$ and

$$g(r) = \mu + 2|c|(1-\beta)\cos\lambda[k|\cos(\lambda-\phi)|\sin^{-1}r + |\sin(\lambda-\phi)| \log \frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}] < \pi, \quad (0 \leq r < 1). \quad \dots(3)$$

Then $\operatorname{Re} \left(1 - \frac{1}{c} + \frac{z}{c} \frac{F'(z)}{F(z)} \right) > 0$ in $|z| < R_0$ where R_0 is the least positive root of the equation

$$h(r) = 1 + r^2[2(1-\beta)\cos^2\lambda - 1] - kr(1-\beta) \cos\lambda \sec[g(r)/2] = 0. \quad \dots(4)$$

Theorem 2—Let F be as in Theorem 1 with f_1, \dots, f_n in $V_k^\lambda(\beta, c)$. Let $g(r)$ be as in (3). Then $\operatorname{Re} \left(1 + \frac{z}{c} \frac{F''(z)}{F'(z)} \right) > 0$ in $|z| < R_0$ where R_0 is as above.

The above two Theorems unify and generalize many results found in literature. In fact the following corollaries are immediate.

Corollary 1—The conditions on $\gamma_1, \dots, \gamma_n$ being as in Theorem 1 a region $|z| < R_0$ of starlikeness of $\gamma_1 f_1 + \dots + \gamma_n f_n$, where f_1, \dots, f_n are λ -spirallike of order β , is obtained by putting $c = 1$ and $k = 2$ in Theorem 1.

As a further special case of this Corollary (when $\lambda = 0$) we obtain a region of starlikeness of linear combinations of starlike functions of order β .

Corollary 2—Let $n = 2$ and the conditions on γ_1 and γ_2 be as in Theorem 1. Then a region $|z| < R_0$ of starlikeness of $\gamma_1 f_1 + \gamma_2 f_2$ where f_1, f_2 are functions of bounded radius rotation is obtained by putting $\lambda = 0 = \beta$ and $c = 1$ in Theorem 1. We thus obtain Theorem 4 in Padmanabhan and Parvatham¹⁰.

Corollary 3—The conditions on $\gamma_1, \dots, \gamma_n$ being as in Theorem 1 a region $|z| < R_0$ of convexity of $\gamma_1 f_1 + \dots + \gamma_n f_n$ where f_1, \dots, f_n are convex functions of order β , is obtained by putting $c = 1, k = 2$ and $\lambda = 0$ in Theorem 2. We thus

obtain Theorem 2 in Bhargava and Rao². In particular when $n = 2$ we get Theorem 2 of Silverman and Silvia¹⁶.

Next, a region of convexity of $F = \gamma_1 f_1 + \dots + \gamma_n f_n$ is obtained when each f_j is convex, by putting $\beta = 0$ in the above. In particular when $n = 2$ this yields Theorem 1 of Stump¹⁷ which in turn contains the results of MacGregor⁶ and Labelle and Rahman⁴.

Corollary 4—The conditions on $\gamma_1, \dots, \gamma_n$ being as in Theorem 1, a region $|z| < R_0$ of convexity of $\gamma_1 f_1 + \dots + \gamma_n f_n$ where zf'_1, \dots, zf'_n are λ -spirallike functions of order β is obtained by putting $c = 1$ and $k = 2$ in Theorem 2.

Corollary 5—The conditions on $\gamma_1, \dots, \gamma_n$ being as in Theorem 1, a region $|z| < R_0$ of convexity of $\gamma_1 f_1 + \dots + \gamma_n f_n$ where f_1, \dots, f_n are functions of bounded boundary rotation is obtained by putting $c = 1$ and $\lambda = 0 = \beta$. Again when $n = 2$ and $k = 2$ this reduces to Theorem 1 in Stump¹⁷.

Lemma 1 stated below in Section 2 and proved by the authors² is a key for all our discussions. It provides a direct generalization of a Lemma Stump¹⁷. Stump devised his Lemma for discussing linear combinations $\gamma_1 f_1 + \gamma_2 f_2$ of two convex functions f_1 and f_2 . Padmanabhan and Parvatham¹⁰ have used Stump's Lemma while discussing $\gamma_1 f_1 + \gamma_2 f_2$ when f_1 and f_2 are in the class U_k . Campbell³ has given an excellent treatment of various radii results for linear combinations of n functions in various classes of analytic functions. However, Campbell has not considered the spaces considered by us here. Moreover, unlike us, Campbell felt that Stump's formulation of determining the radii results for $\gamma_1 f_1 + \gamma_2 f_2$ in terms of the joint parameter $\alpha = \arg(\gamma_1/\gamma_2)$ discouraged a generalization of the problem to arbitrary finite combinations $\sum_{j=1}^n \gamma_j f_j$ ($\sum_{j=1}^n \gamma_j = 1$) and thus he reformulated the problem in terms of the bound on the parameters $\arg \gamma_j$.

2. SOME PRELIMINARY RESULTS

Lemma 1—If a, u_j and $\beta_j \neq 0$ ($j = 1, \dots, n$) are complex numbers with $\sum_{j=1}^n \beta_j = 1$ and $d \geq 0$ such that $|u_j - a| \leq d$ ($j = 1, \dots, n$) and $0 \leq \theta = \max_{1 \leq i, j \leq n} \arg(\beta_i/\beta_j) < \pi$, then

$$\operatorname{Re} \sum_{j=1}^n u_j \beta_j \geq \operatorname{Re} a - d \sec \theta/2.$$

PROOF: See Bhargava and Rao².

Lemma 2—Let $g(r)$ be defined by

$$g(r) = \mu + A_0 \sin^{-1} r + A_1 \log \frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}$$

where $0 \leq r < 1$, $A_0 > 0$, $A_1 \geq 0$, $\mu < \pi$ and $p \geq 0$. Then $g(r)$ increases strictly in $[0, 1)$ from μ to ∞ and therefore there exists a unique r_0 in $(0, 1)$ such that $g(r_0) = \pi$.

PROOF : It is easy to see that $g(0) = \mu$ and $g(r) \rightarrow \infty$ as $r \rightarrow 1 - 0$.

Now

$$g'(r) = \frac{A_0}{\sqrt{1-r^2}} + \frac{A_1(k+2r)}{1-r^2} > 0$$

and hence the result is true.

Lemma 3—If f is in $U_k^\lambda(\beta, c)$ and $\phi = \arg c$ then

$$|\arg f/z| \leq |c| (1-\beta) \cos \lambda [k |\cos(\lambda - \phi)| \sin^{-1} r + |\sin(\lambda - \phi)| \log \frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}], \quad (0 \leq r = |z| < 1).$$

PROOF : Since f is in $U_k^\lambda(\beta, c)$, we may choose g in U_k such that¹,

$$(f/z) \frac{e^{i\lambda} \sec \lambda}{c(1-\beta)} = g/z.$$

Here using¹,

$$|\arg g/z| \leq k \sin^{-1} r, \quad \log |g/z| \leq \log \frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}$$

we have the required inequality.

3. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1— $F(z) = \gamma_1 f_1(z) + \dots + \gamma_n f_n(z)$ where f_j are in $U_k^\lambda(\beta, c)$. Hence

$$1 - \frac{1}{c} + \frac{z}{c} \frac{F'(z)}{F(z)} = \sum_1^n u_j(z) \beta_j(z).$$

where

$$u_j(z) = 1 - \frac{1}{c} + \frac{zf'_j(z)}{cz_j(z)} \quad \text{and} \quad \beta_j(z) = \frac{\gamma_j f_j(z)}{\sum_1^n \gamma_j f_j(z)}.$$

Since each f_j is in $U_k^\lambda(\beta, c)$, we have from Bhargava and Nanjunda Rao¹ $|u_j - a| \leq d$ where

$$a = \frac{1 + r^2 (2e^{-i\lambda} (1-\beta) \cos \lambda - 1)}{1 - r^2}, \quad d = \frac{kr(1-\beta) \cos \lambda}{1 - r^2}.$$

Hence from Lemma 1, $\operatorname{Re} \left(1 - \frac{1}{c} + \frac{z}{c} \frac{F'}{F} \right) > 0$ for $|z| < R_0$ where R_0 is the least positive root of the equation $\operatorname{Re} a - d \sec g(r)/2 = 0$, that is, R_0 is the least positive root of the equation $h(r) = 0$. That $h(r) = 0$ has a positive root follows since $h(0) = 1$ and $h(r) \rightarrow -\infty$, since $g(r) \rightarrow \pi$ (as $r \rightarrow r_0$ where r_0 is as in Lemma 2).

Proof of Theorem 2—Since f_j is in $V_k^\lambda(\beta, c)$ we see that zf_j' is in $U_k^\lambda(\beta, c)$ (Bhargava and Rao⁷). The Theorem now follows immediately from Theorem 1 on changing f_j to zf_j' there.

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CONVOLUTIONS OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Schild and Silverman² have investigated some properties of convolutions of univalent functions with negative coefficients which are starlike of order α or convex of order α . Reddy and Padmanabhan¹ have examined a certain class of p -valent starlike functions $S_p(A, B)$. Using their results, we investigate some properties of convolutions of univalent functions with negative coefficients from either a class $T^*(A, B)$ or $C(A, B)$.

1. INTRODUCTION

Let S denote the class of normalized univalent functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in the unit disc $E = \{z: |z| < 1\}$. We denote by T the subclass of functions of S of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. $S^*(\alpha)$ and $K(\alpha)$ denote respectively subclasses of S containing starlike and convex functions of order α , $0 \leq \alpha < 1$. We denote by $T^*(\alpha)$ and $C(\alpha)$, the subclasses of T which are respectively starlike of order α and convex of order α . These classes were introduced by Silverman³.

Let $H = \{w: \text{analytic in } E, w(0) = 0, |w(z)| < 1 \text{ in } E\}$. Let $P(A, B)$ denote the class of analytic functions in E which are of the form $\frac{1+Aw(z)}{1+Bw(z)}$, $-1 \leq A < B \leq 1$, $w \in H$.

Define

$$S^*(A, B) = \{f, f \in S \text{ and } \frac{zf'}{f} \in P(A, B)\}$$

and

$$K(A, B) = \{f, f \in S \text{ and } \frac{(zf')'}{f'} \in P(A, B)\}.$$

We further define $T^*(A, B) = \{f, f \in T \text{ and } \frac{zf'}{f} \in P(A, B)\}$ and $C(A, B) = \left\{f: f \in T \text{ and } \frac{(zf')'}{f'} \in P(A, B)\right\}.$

If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ with $a_n, b_n \geq 0$, their

Hadamard product or convolution is defined by $h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$

In a recent paper Schild and Silverman² investigated some properties of convolutions of functions with negative coefficients from $T^*(\alpha)$ or $C(\alpha)$. It is our aim in this paper to investigate the corresponding properties of the convolutions of functions from the classes $T^*(A, B)$ or $C(A, B)$.

Again Reddy and Padmanabhan¹ examined p -valent starlike functions with negative coefficients in E . In the sequel we make frequent use of the following lemmas proved by Reddy and Padmanabhan¹.

Lemma A—A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ is in $T^*(A, B)$ if and only

if

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m \leq 1.$$

Lemma B—A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ is in $C(A, B)$ if and only

if

$$\sum_{m=2}^{\infty} m \left\{ \frac{m(B+1) - (A+1)}{B-A} \right\} a_m \leq 1.$$

This follows from Lemma A, on observing that

$$f \in C(A, B) \Leftrightarrow zf' \in T^*(A, B).$$

2. CONVOLUTIONS OF FUNCTIONS FROM SUBCLASSES OF $T^*(A, B)$

Let us now investigate the nature of $h(z) = f(z) * g(z)$, given that $f(z)$ and $g(z)$ belong to $T^*(A, B)$ and $C(A, B)$.

Theorem 1—If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $a_n \geq 0$; $b_n \geq 0$ are elements of $T^*(A, B)$ then

$h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ is an elements of $T^*(A_1, B_1)$ with $-1 \leq A_1 < B_1 \leq 1$ where $A_1 \leq 1 - 2k$, $B_1 > \frac{A_1 + k}{1 - k}$ and that the bounds for A_1 and B_1 cannot be improved.

PROOF : From Lemma A, we know that

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m \leq 1 \quad \dots(1)$$

and

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} b_m \leq 1. \quad \dots(2)$$

We wish to find values A_1, B_1 such that $-1 \leq A_1 < B_1 \leq +1$, v for $h(z) = f(z) * g(z) \in T^*(A_1, B_1)$. Equivalently we want to determine A_1, B_1 satisfying

$$\sum_{m=2}^{\infty} \frac{m(B_1+1) - (A_1+1)}{B_1 - A_1} a_m b_m \leq 1. \quad \dots(3)$$

Combining (1) and (2), we get using the Cauchy-Schwarz inequality

$$\sum_{m=2}^{\infty} u \sqrt{a_m b_m} \leq \left(\sum_{m=2}^{\infty} u a_m \right)^{1/2} \left(\sum_{m=2}^{\infty} u b_m \right)^{1/2} \leq 1 \quad \dots(4)$$

where

$$u = \frac{m(B+1) - (A+1)}{(B-A)}.$$

(3) is satisfied if

$$u_1 a_m b_m \leq u \sqrt{a_m b_m}$$

where

$$u_1 = \frac{m(B_1+1) - (A_1+1)}{B_1 - A_1} \quad \dots(5)$$

for $m = 2, 3, \dots$, that is, if $u_1 \sqrt{a_m b_m} \leq u$.

But from (4) we have

$$\sqrt{a_m b_m} \leq \frac{1}{u}, \quad m = 2, 3, \dots \quad \dots(6)$$

Therefore it is enough to find u_1 such that

$$\frac{1}{u} \leq \frac{u}{u_1}$$

or

$$u_1 \leq u^2 \quad \dots (7)$$

(7) is equivalent to

$$\frac{m(B_1 + 1) - (A_1 + 1)}{(B_1 - A_1)} \leq \left(\frac{m(B + 1) - (A + 1)}{B - A} \right)^2 = u^2, m \geq 2.$$

That is, $m(B_1 + 1) - (A_1 + 1) \leq u^2(B_1 - A_1)$.

This yields

$$A_1 < \frac{u^2 B_1 - m(B_1 + 1) + 1}{u^2 - 1} \quad \dots (8)$$

It is easy to verify that $u^2 > 1$ for $m \geq 2$. Now (8) gives on simplification.

$$\frac{B_1 - A_1}{B_1 + 1} \leq \frac{m - 1}{u^2 - 1}, \text{ for } m \geq 2. \quad \dots (9)$$

The right hand member decreases as m increases and so is maximum for $m = 2$. So (9) is satisfied provided

$$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{(B - A)^2}{(2B - A + 1)^2 - (B - A)^2} = k. \quad \dots (10)$$

say

Obviously $k < 1$ and fixing A_1 in (10), we get

$$B_1 \geq \frac{k + A_1}{1 - k}. \quad \dots (11)$$

It is easy to verify that $-1 < A_1 < B_1 \leq 1$. When we take

$$f(z) = g(z) = z - \frac{(B - A)}{(2B - A + 1)} z^2 \in T^*(A, B),$$

it follows that

$$h(z) = f(z) * g(z) = z - \frac{(B - A)^2}{(2B - A + 1)^2} z^2.$$

Then

$$\frac{2(B_1 + 1) - (A_1 + 1)}{(B_1 - A_1)} = \frac{(2B - A + 1)^2}{(B - A)^2},$$

showing that $h \in T^*(1 - 2k, 1)$, with k as in (10).

Remarks : Taking $A = 2\alpha - 1$, $B = 1$, the class $T^*(A, B)$ reduces to $T^*(\alpha)$ and our results yield the corresponding result in Schild and Silverman² on the convolution of two functions in $T^*(\alpha)$.

Corollary—For $f(z)$ and $g(z)$ as in Theorem 1, the function h given by $h(z) = z - \sum_{n=2}^{\infty} \sqrt{a_n b_n} z^n \in T^*(A, B)$. The result follows immediately from (4) using the Cauchy-Schwarz inequality. For the same functions as in Theorem 1, the result is best possible.

Theorem 2—If $f \in T^*(A, B)$ and $g \in T^*(A', B')$ then $f * g \in T^*(A_1, B_1)$ where $A_1 \leq 1 - 2k$ and $B_1 \geq \frac{A_1 + k}{1 - k}$ with

$$k = \frac{(B - A)(B' - A')}{(2B - A + 1)(2B' - A' + 1) - (B - A)(B' - A')}.$$

The result is best possible.

PROOF : Proceeding exactly as in Theorem 1, we require

$$\begin{aligned} \frac{m(B_1 + 1) - (A_1 + 1)}{(B_1 - A_1)} &\leq \frac{m(B + 1) - (A + 1)}{(B - A)} \\ &\times \frac{m(B' + 1) - (A' + 1)}{B' - A'} = C, \\ &\text{for all } m \geq 2. \end{aligned}$$

That is

$$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{m - 1}{C - 1}. \quad \dots(12)$$

The function $\frac{m - 1}{C - 1}$ is decreasing with respect to m and is maximum for $m = 2$, we get from (12),

$$\frac{B_1 - A_1}{B_1 + 1} \geq \frac{(B - A)(B' - A')}{(2B - A + 1)(2B' - A' + 1) - (B - A)(B' - A')} = k. \quad \dots(13)$$

Clearly $k < 1$.

Fixing A_1 in (13) we get $B_1 \geq \frac{A_1 + k}{1 - k}$. As we require $B_1 \leq 1$, we immediately obtain $A_1 \leq 1 - 2k$. If we now take

$$f(z) = z - \frac{B - A}{2B - A + 1} z^2 \in T^*(A, B)$$

and

$$g(z) = z - \frac{B' - A'}{2B' - A' + 1} z^2 \in T^*(A', B')$$

we see that

$$f * g(z) = z - \frac{(B-A)(B'-A')z^2}{(2B-A+1)(2B'-A'+1)}.$$

Then

$$\frac{2(B_1+1)-(A_1+1)}{B_1-A_1} = \frac{(2B-A+1)(2B'-A'+1)}{(B-A)(B'-A')}$$

when $A_1 = 1 - 2k$ and $B_1 = 1$ showing that our result is best possible.

Remark : Again putting $A = 2\alpha - 1$, $B = 1$, $A' = 2\gamma - 1$ and $B' = 1$ in Theorem 2 we get $f(z) \in T^*(\alpha)$ and $g(z) \in T^*(\gamma)$ and the corresponding result that $f(z) * g(z) \in T^*\left(\frac{2-\alpha\gamma}{2-\alpha-\gamma}\right)$ as in Schild and Silverman² immediately follows.

Corollary—If $f(z), g(z), h(z) \in T^*(A, B)$ then $f(z) * g(z) * h(z) \in T^*(A_2, B_2)$ where $A_2 \leq 1 - 2k_1$

$$B_2 \geq \frac{A_2 + k_1}{1 - k_1} \text{ with } k_1 = \frac{(B-A)(B_1-A_1)}{(2B_1-A_1+1)(2B-A+1) - (B-A)(B_1-A_1)}$$

where A_1, B_1 are given as in Theorem 1.

PROOF : $f(z), g(z) \in T^*(A, B)$. Therefore by Theorem 1, $F(z) = f(z) * g(z) \in T^*(A_1, B_1)$ where $A_1 \leq 1 - 2k$ and $B_1 \geq \frac{A_1 + k}{1 - k}$ with $k = \frac{(B-A)^2}{(2B-A+1)^2 - (B-A)^2}$. Now $F(z) \in T^*(A_1, B_1)$ and $h(z) \in T^*(A, B)$ and by Theorem 2, the result follows. For functions of the class $C(A, B)$ we have the following similar results. Using Lemma B and Proceeding exactly as in Theorem 1, we get :

Theorem 3—If $f \in C(A, B)$ and $g \in C(A', B')$ then $f * g \in C(A_1, B_1)$ where $A_1 \leq 1 - 2k$ and $B_1 \geq \frac{A_1 + k}{1 - k}$ with

$$k = \frac{(B-A)(B'-A')}{(2B-A+1)(2B'-A'+1) - (B-A)(B'-A')}.$$

The result is best possible.

Remarks : (1) Choosing $f(z) = z - \frac{(B-A)}{2(2B-A+1)}z^2 \in C(A, B)$ and $g(z) = z - \frac{(B'-A')}{4(2B-A+1)(2B'-A'+1)}z^2 \in C(A_1, B_1)$ where $A_1 = 1 - 2k$ and $B_1 = 1$ with k as given above. We can show that our estimates are sharp, in Theorem 3.

(2) Putting $A = 2\alpha - 1$, $A' = 2\gamma - 1$, $B = B' = 1$ we get the corresponding result of Silverman and Schild² for the convolution of two functions from $C(\alpha)$ and $C(\gamma)$. It follows from Theorem 1 that if $f, g \in T^*(-1, +1)$ then $f * g \in T^*(A_1, B_1)$ where $A_1 \leq \frac{1}{3}$ and $B_1 \geq \frac{3A_1 + 1}{2}$.

Using both Lemmas A and B and proceeding exactly as in Theorem 1, we get the following interesting result.

Theorem 4—If $f \in T^*(A, B)$ and $g \in T^*(A', B')$ then $h(z) = f(z)^* g(z) \in C(A_1, B_1)$ where $A_1 \leq 1 - 2k$, $B_1 \geq \frac{A_1 + k}{1 - k}$ with

$$k = \frac{2(B-A)(B'-A')}{(2B-A+1)(2B'-A'+1) - 2(B-A)(B'-A')}.$$

The result is best possible.

Remarks : (1) The functions $f(z) = z - \frac{B-A}{2B-A+1} z^2 \in T^*(A, B)$ and $g(z) = z - \frac{B'-A'}{2B'-A'+1} z^2 \in T^*(A', B')$ clearly show that our bounds for B_1 and A_1 are the best possible with k given above. Here $f^* g \in C(1 - 2k, 1)$.

(2) On putting $B = B' = 1$ and $A = 2\alpha - 1$, $A' = 2\gamma - 1$, Theorem 4 immediately reduces the corresponding result of Silverman and Schild².

Theorem 5—If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ belongs to $T^*(A, B)$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ with $|b_i| \leq 1$, $i \geq 2$, then $f^* g \in S^*(A, B)$.

PROOF : Since $f \in T^*(A, B)$, we have

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m \leq 1. \text{ Further } |b_i| \leq 1, i \geq 2.$$

Therefore

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m b_m &\leq \sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m |b_m| \\ &\leq 1. \end{aligned}$$

This shows that $f(z)^* g(z) = z - \sum_{m=2}^{\infty} a_m b_m z^m \in S^*(A, B)$.

Corollary—If $f(z) \in T^*(A, B)$ and $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $0 \leq b_i \leq 1$ for $i \geq 2$, then $f^* g \in T^*(A, B)$.

Note : $g(z)$ need not even be univalent.

In an exactly similar manner we have :

Theorem 6—If $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$ belongs to $C(A, B)$ and $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $|b_i| \leq 1$ for $i \geq 2$, then $f * g \in K(A, B)$.

Corollary—If $f(z) \in C(A, B)$ and $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $0 \leq b_i < 1$, $i \geq 2$ then $f * g \in C(A, B)$.

Remark : Our results generalize the corresponding results for $T^*(\alpha)$ and $C(\alpha)$ as in Schild and Silverman².

Consider the functions $f(z) = z - \frac{B-A}{2B-A+1} z^2$ and $g(z) = z - \frac{(B-A)}{(3B-A+2)} z^3$ in $T^*(A, B)$. It is clear that $h(z) = z - \frac{(B-A)}{(2B-A+1)} z^2 - \frac{(B-A)}{(3B-A+2)} z^3$ belongs to $T^*(-1, +1)$ only for values of A and B such that $3(B-A)^2 \leq (A+1)^2$, and $h(z)$ need not belong to $T^*(-1, +1)$ for all A and B . In other words, $f, g \in T^*(A, B)$ need not necessarily imply that $h(z) = z - \sum_{n=2}^{\infty} (a_n + b_n) z^n \in T^*(A_1, B_1)$ for any pair of values A_1, B_1 . But we have :

Theorem 7—If $f, g \in T^*(A, B)$ then $h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \in T^*(A_1, B_1)$

where $A_1 \leq 1 - 2k$ and $B_1 \geq \frac{A_1 + k}{1 - k}$ with

$$k = \frac{2(B-A)^2}{(2B-A+1) - 2(B-A)}.$$

Our result is best possible.

PROOF : Since $f, g \in T^*(A, B)$,

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m \leq 1$$

and

$$\sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} b_m \leq 1.$$

Now

$$\sum_{m=2}^{\infty} \left\{ \frac{m(B+1) - (A+1)}{B-A} a_m \right\}^2 \leq \left\{ \sum_{m=2}^{\infty} \frac{m(B+1) - (A+1)}{B-A} a_m \right\}^2 \leq 1. \quad \dots(14)$$

Similarly

$$\sum_{m=2}^{\infty} \left\{ \frac{m(B+1) - (A+1)}{B-A} b_m \right\}^2 \leq 1. \quad \dots(15)$$

Hence

$$\sum_{m=2}^{\infty} \frac{1}{2} \left(\frac{m(B+1) - (A+1)}{B-A} \right)^2 (a_m^2 + b_m^2) \leq 1. \quad \dots(16)$$

$h(z) \in T^*(A_1, B_1)$ if and only if

$$\sum_{m=2}^{\infty} \left(\frac{m(B_1+1) - (A_1+1)}{B_1-A_1} \right) (a_m^2 + b_m^2) \leq 1. \quad \dots(17)$$

Comparing (17) with (16) we see that (17) is true if

$$\frac{m(B_1+1) - (A_1+1)}{(B_1-A_1)} \leq \frac{1}{2} \left(\frac{m(B+1) - (A+1)}{B-A} \right)^2 = \frac{u^2}{2}. \quad \dots(18)$$

Simplifying (18) we get,

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(m-1)}{u^2-2} = y(m). \quad \dots(19)$$

Since $y(m)$ is a decreasing function of m and on putting $m = 2$ in (19), we get

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(B-A)^2}{(2B-A+1)^2 - 2(B-A)^2} = k. \quad \dots(20)$$

Keeping A_1 fixed in (20) we get $B_1 \geq \frac{A_1+k}{1-k}$ and $B_1 \leq 1$ gives $A_1 \leq 1-2k$ with k given as in (20).

The functions $f(z) = g(z) = z - \frac{B-A}{(2B-A+1)} z^2$ show that our result is sharp.

Actually, $h(z) = z - \frac{2(B-A)^2}{(2B-A+1)^2} z^2 \in T^*(1-2k, 1)$ with k as in (20).

Remark : Again with $A = 2\alpha - 1$ and $B = 1$ we get the results of Schild and Silverman as in².

We have shown in Theorem 1 that if $f, g \in T^*(A, B)$, then $f * g \in T^*(A_1, B_1)$ where $A_1 \leq 1-2k$ and $B_1 \geq \frac{A_1+k}{1-k}$ with $k = \frac{(B-A)^2}{(2B-A+1)^2 - (B-A)^2}$. Conversely if $h(z) \in T^*(A_1, B_1)$, A_1, B_1 as described above, do there exist functions $f, g \in T^*(A, B)$ such that $h(z) = f(z) * g(z)$? The answer is negative as is shown by the following example.

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ with $f, g \in T^*(A, B)$.

Clearly

$$a_m \leq \frac{B-A}{m(B+1) - (A+1)} \text{ and } b_m \leq \frac{B-A}{m(B+1) - (A+1)}, \quad m \geq 2.$$

Hence

$$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \in T^*(A_1, B_1)$$

as described in

Theorem 1—But $a_m b_m \leq \frac{(B-A)^2}{(m(B+1) - (A+1))^2}$, $m \geq 2$ for the convolution of any two functions in $T^*(A, B)$. Now consider :

$$h(z) = z - \frac{B_1 - A_1}{m(B_1+1) - (A_1+1)} z^m \in T^*(A_1, B_1).$$

For $h(z)$ we have,

$$\frac{B_1 - A_1}{m(B_1+1) - (A_1+1)} > \frac{(B-A)^2}{(m(B+1) - (A+1))^2}$$

for $m \geq 2$ as in Theorem 1. This shows that there are no f and g in $T^*(A, B)$ for which $h = f * g$, though $h \in T^*(A_1, B_1)$.

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2. THE EQUATIONS OF MOTION

To simplify the problem, we average the problem by the Gaussian scheme over the fast variable l, l', g_0 and g_1 , also assuming that a/a' is small value. Hence, the Hamiltonian of the problem can be written in the form¹¹

$$\bar{F} = K_0 + K_1 \quad \dots(1)$$

where K_0 is considered the unperturbed function while K_1 is the perturbed function

$$\begin{aligned} K_0 = & -\frac{\mu^2 m^3}{2L^2} + \frac{1}{2} \left(\frac{G_0^2}{A_0} - \frac{G_1^2}{A_1} \right) - \frac{k^2 m_2 m}{a'} [X_{0,0}^{0,0}(e) C_{0,0,0}^{0,\frac{1}{2}}(I)] \\ & - Q_0 [(3 \cos^2 I - 1)(3 \cos^2 I_0 - 1) + 3 \sin 2I \sin 2I_0 \cos(h_0 - h) \\ & + 3 \sin^2 I \sin^2 I_0 \cos 2(h_0 - h)] - Q_1 [(3 \cos^2 I - 1)(3 \cos^2 I_1 - 1) \\ & + 3 \sin 2I \sin 2I_1 \cos(h_1 - h) + 3 \sin^2 I \sin^2 I_1 \cos 2(h_1 - h)] \\ K_1 = & -k^2 m_2 m \frac{a^2}{a'^3} \left\{ \left(1 + \frac{3}{2} e^2 \right) C_{0,0,0}^{2,\frac{1}{2}}(I) + \frac{5}{2} e \left[C_{1,1,0}^{2,\frac{1}{2}}(I) \right. \right. \\ & \left. \left. + C_{-1,-1,0}^{2,\frac{1}{2}}(I) \right] \cos 2g \right\} \\ Q_0 = & \frac{1}{16} m_0 k^2 \frac{(C_0 - A_0)}{(1 - e^2)^{3/2}} (3 \cos^2 J_0 - 1) \\ Q_1 = & \frac{1}{16} m_0 k^2 \frac{(C_1 - A_1)}{(1 - e^2)^{3/2}} (3 \cos^2 J_1 - 1) \end{aligned}$$

and

$$C_{0,0,0}^{0,\frac{1}{2}}, C_{0,0,0}^{2,\frac{1}{2}}, C_{1,1,0}^{2,\frac{1}{2}}, C_{-1,-1,0}^{2,\frac{1}{2}}$$

are defined in Brumberg².

The equations of the unperturbed motion of M_0, M_1 and M_2 in the canonical form

$$\begin{aligned} dl/dt &= \partial K_0 / \partial L, \quad dL/dt = -\partial K_0 / \partial l \\ dl_i/dt &= \partial K_0 / \partial L_i, \quad dL_i/dt = -\partial K_0 / \partial l_i \quad (i = 0, 1). \end{aligned} \quad \dots(2)$$

3. THE SOLUTION OF UNPERTURBED EQUATIONS

The equation of unperturbed motion (2) gives the first integrals

$$\begin{aligned} K_0 &= \text{const.}, \quad L = \text{const.}, \quad G = \text{const.}, \\ L_i &= \text{const.}, \quad G_i = \text{const.} \end{aligned} \quad \dots(3)$$

$$\begin{aligned} \psi_1 = & bA \left(1 - \frac{\omega}{u} \right) + 2cAH_0^0 - 2dAH_1^0 \frac{\omega}{u} + e^* A \left(H_1^0 - H_0^0 \frac{\omega}{u} \right) \\ & + f \left(H_0^{0^2} A \frac{\omega}{u} + 2H_0^0 H_1^0 - A^2 B \frac{\omega}{u} + H_1^{0^2} A \right. \\ & \left. - 2H_0^0 H_1^0 A \frac{\omega}{u} - A^2 B \frac{\omega^2}{u^2} \right) \\ & \dots \dots \dots \end{aligned}$$

The constants are easily obtained from Sidlichovsky¹¹.

We note that in the unperturbed motion, the semimajor axis and the eccentricity of orbit M_1 remain constant while its inclination angle and the projections H_0, H_1 of the rotational momentum vectors into the direction of the total angular momentum vector of the system are harmonic or exponential function of the time parameter τ .

4. THE PERTURBED MOTION

We note from the perturbed Hamiltonian K_1 that the elements of translational motion l, g, h and G are the only quantities which may be possibly affected by the motion of M_2 in a circle. The perturbation of these elements can be obtained from the following equations :

$$\begin{aligned} \delta l &= \int_{t_0}^t (\partial K_1 / \partial L)_0 dt, \quad \delta g = \int_{t_0}^t (\partial K_1 / \partial G)_0 dt \\ \delta h &= \int_{t_0}^t (\partial K_1 / \partial H)_0 dt, \quad \delta G = - \int_{t_0}^t (\partial K_1 / \partial g)_0 dt \end{aligned} \quad \dots(9)$$

where the subscript 0 refers to the values calculated in the unperturbed motion.

Thus, the motion of the body M_2 affects only the eccentricity and the inclination orbital plane of the axisymmetric rigid body M_1 relative to the other axisymmetric M_0 . The other parameter involved in the problem is found to be identical with those of unperturbed motion.

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A NOTE ON THE MINIMUM POTENTIAL STRENGTH

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Recently, an estimate for the minimum potential strength which produces bound states was deduced, subject to a sufficiency condition, the interesting thing being that most potentials of physical interest satisfy this condition. The estimate is now vastly improved upon : The sufficiency condition is incorporated into a well known iterative process to reduce an otherwise unwieldy but better estimate to a simple and useful one. It is then verified that the new, but equally simple estimate actually gives better results than the previous one, for some important potentials.

1. INTRODUCTION

Recently it was shown¹ that λ_1 , an upper bound for the minimum potential strength λ_0 which produces bound states, coincides with λ_0 when a certain sufficiency condition is satisfied. That is

$$\lambda_0 \leq \lambda_1 \equiv \frac{\int_0^{\infty} r^2 U(r) dr}{\int_0^{\infty} r U(r) A(r) dr} \quad \dots(1)$$

where,

$$A(r) \equiv \int_0^{\infty} G^{(0)}(r, r') U(r') r' dr'$$

$$G^{(0)}(r, r') = -r, r' > r$$

$$= -r', r' < r$$

but

...(2)

$$\lambda_0 = \lambda_1$$

...(3)

if

$$b \equiv D\lambda_1^2 = 1, \quad D \equiv \frac{\int_0^\infty U(r) [A(r)]^2 dr}{\int_0^\infty U(r) r^2 dr}. \quad \dots(4)$$

In the process condition (4) was shown to ensure the following :

$$A(r) \propto r$$

$$\text{wave function} \propto r. \quad \dots(5)$$

If the condition (4) is only approximately satisfied, then equations (3) and (5) become approximate equalities. The important thing is that this is the case for a large number of potentials of physical interest².

We will now incorporate the sufficiency parameter b as defined in (4) into an iterative process to considerably improve upon the previous result. We will deduce a more accurate and at the same time an equally simple estimate, namely

$$\lambda_0 \approx \lambda_1/b. \quad \dots(6)$$

2. THE MINIMUM POTENTIAL STRENGTH

We first recapitulate the following well known results³ :

(i) The minimum potential strength λ_0 is the smallest positive eigen value of the homogeneous integral equation,

$$\psi(r) = \lambda \int_0^\infty G^{(0)}(r, r') U(r') \psi(r') dr'. \quad \dots(7)$$

(ii) This smallest eigen value has an upper bound given by the expression,

$$\lambda_0 \leq \lambda_2 \equiv \frac{\int_0^\infty \phi^2(r) U(r) dr}{\int_0^\infty \int_0^\infty \phi(r) U(r) G^{(0)}(r, r') U(r') \phi(r') dr' dr} \quad \dots(8)$$

where $\phi(r)$ is an arbitrary function. However, the closer the function $\phi(r)$ is to the actual solution of (7), the closer is the upper bound; when $\phi(r)$ is the exact solution of (7), the upper bound λ_2 coincides with λ_0 . This can be easily verified by using eqn. (7) in (8). In fact one could build up an iterative procedure in the usual way.

We first take $\phi(r) = \psi^{(n)}(r)$ where $\psi^{(n)}$ is some suitable trial function. Substitution of $\psi^{(n)}$ in the inequality (8) gives an upper bound for λ_0 . We next obtain a better

approximation to $\psi(r)$, namely $\psi^{(1)}$ by substituting $\psi^{(0)}$ in the right side of equation (7). Substitution of $\psi^{(1)}$ thus obtained in inequality (8) gives an even closer upper bound to λ_0 , and so on.

We also observe that the bound λ_1 , given in (1) can be recovered from the upper bound λ_2 given in (8) by the following method.

We use the fact that at the minimum potential strength, $l = 0$, $k^2 = 0$ and $\delta_0 = \pi/2$ where δ_0 is the $l = 0$ zero energy resonance phase shift. So the radial Schrodinger equation

$$u'' + \left[k^2 - \frac{l(l+1)}{r^2} - \lambda_0 U(r) \right] u = 0$$

and its equivalent integral form⁴, namely

$$u(r) = r \cos \delta_l j_l(kr) + kr n_l(kr) \int_0^r j_l(kr') U(r') u(r') r' dr' \\ + kr j_l(kr) \int_r^\infty n_l(kr') U(r') u(r') r' dr'$$

go over respectively into

$$u'' - \lambda_0 U(r) u = 0 \quad \dots(9)$$

and eqn. (7). As can be seen from (9), the zeroth approximation obtained by putting $U(r) = 0$, that is the free wave function, is given by the equation,

$$u^{(0)''} = 0.$$

That is

$$\psi^{(0)}(r) = u^{(0)}(r) = vr \quad \dots(10)$$

where v is a constant, because of the boundary condition $u(0) = 0$. Substituting equation (10) into (8) we recover (1). We now use the iterative procedure described earlier. Accordingly, in the integral on the right side of equation (7), we replace $\psi(r)$ by its zeroth approximation, given in (10).

This gives

$$\psi^{(1)}(r) \approx \lambda_1 v A(r). \quad \dots(11)$$

The substitution of (11) in (8) gives an even closer upper bound than λ_1 (which is obtained by substituting (10) in (8)). That is, if condition (4) is approximately satisfied, we get an even better estimate than (1).

This estimate is

$$\lambda_0 \approx \lambda_3 = \frac{\int_0^\infty U(r) [A(r)]^2 dr}{\int_0^\infty \int_0^\infty A(r) U(r) G^{(0)}(r, r') U(r') A(r') r' dr' dr} \quad \dots(12)$$

For practical purposes we considerably simplify the complicated expression (12) and at the same time we get very good results as follows :

We exploit the fact that if $b \approx 1$, that is, if the condition (4) is approximately satisfied then according to (5) $A(r) \approx \mu$, where μ is a constant; Accordingly in the integrals in the numerator and denominator of (12) we replace 'one of the factors' $A(r)$ by μr (If we substitute in (12) $[A(r)]^2 \approx \mu^2 r^2$, then we are back with equation (5), and recover λ_1).

This of course is valid only if the condition (4) is approximately satisfied. The unknown constants ν , μ are now eliminated and we get

$$\lambda_0 \approx \lambda_2 \approx \lambda_3 \equiv \frac{\int_0^\infty r U(r) A(r) dr}{\int_0^\infty \int_0^\infty A(r) U(r) G^{(0)}(r, r') U(r') r' dr' dr}$$

which on using the expressions (1), (2) and (4) can be shown to reduce to (6) :

$$\lambda_0 \approx \lambda_3 = \lambda_1/b. \quad \dots(6)$$

Thus λ_3 is a simple and intermediate estimate for λ_0 , intermediate in the sense that it is more accurate than λ_1 but less accurate than the clumsy estimate λ_2 given in (12).

In practice, as can be seen from Table I estimate (6) gives very good results indeed.

TABLE I*
A Comparison of the Minimum Potential Strength Estimates :

Potential	(Exact value)	b	λ_0	λ_3
Square Well	(a)			
$U(r) = -V, r < a$ $= 0, r > a$	$2.46/Va^2$	1.0125	$2.469/Va^2$ (0.37)	$2.5/Va^2$ (1.6)
Exponential	(b)			
$U(r) = -\lambda \exp(-2ar)$	$5.784 a^2$	1.0889	$5.874a^2$ (1.5)	$6.4 a^2$ (10.6)
Yukawa	(c)			
$U(r) = -\frac{\lambda}{r} \exp(-ar)$	1.68	1.1507	1.738 (3.4)	2.0 (19.0)

* Figures in brackets denote percentage of error.

(a) Schiff⁵; (b) Ter Haar⁶; (c) Schey⁷ and Schwartz.

3. CONCLUSION

If $b \approx 1$, where b has been defined in (4) then

$$\lambda_0 \approx \lambda_1/b.$$

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ON THE PROPAGATION OF WAVES IN A NON-HOMOGENEOUS FLUID

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This work deals with the propagation of waves in a non-homogeneous fluid. The fluid consists of two horizontal layers of two different fluids, $Z = 0$ and $Z = h$. Waves are generated by a uniform normal pressure acting on the upper fluid surface ($Z = 0$). Results are obtained for both compressible and incompressible fluids.

INTRODUCTION

The problem of propagation of waves in a non-homogeneous fluid has been studied by several authors¹⁻⁴. In the present work, the problem of the propagation of a wave in a non-homogeneous fluid has been studied when a normal pressure acts on the upper boundary of the fluid. This fluid consists of two horizontal layers of different densities.

BASIC EQUATION AND SOLUTION

Let the fluid under consideration occupy the infinite strip $Z = 0$ and $Z = h$ on the Z axis of cylindrical coordinates. This strip consists of two layers of two fluids. The upper fluid has a density ρ_1 and its sound speed is α_1 while the density and sound speed of the lower fluid are ρ_2 and α_2 respectively.

It is well-known that the potential ϕ satisfies the following wave equation

$$\frac{\partial^2 \phi_i}{\partial r^2} + \frac{\partial^2 \phi_i}{\partial Z^2} + \frac{1}{r} \frac{\partial \phi_i}{\partial r} = \frac{1}{\alpha_i^2} \frac{\partial^2 \phi_i}{\partial t^2}, \quad i = 1, 2. \quad \dots(1.1)$$

The boundary conditions are

$$\text{at } Z = 0, \quad -\infty < r < \infty$$

$$\rho \frac{\partial \phi_i}{\partial t} = \begin{cases} P_0 e^{-mt}, & r < R(t) \\ 0, & r > R(t) \end{cases} \quad \dots(1.2)$$

$$\text{at } Z = h, \quad -\infty < r < \infty$$

$$\left. \begin{aligned} \rho_1 \frac{\partial \phi_1}{\partial t} &= \rho_2 \frac{\partial \phi_2}{\partial t} \\ \frac{\partial \phi_1}{\partial Z} &= \frac{\partial \phi_2}{\partial Z} \end{aligned} \right\} \quad \dots(1.3)$$

where, $R(t)$ is the coordinate of the front of the propagating pressure, P_0 is the pressure on the boundary ($Z = 0$) and m is a constant.

Using Laplace transform, eqn. (1.1) will take the form

$$\frac{\partial^2 \bar{\varphi}_I}{\partial r^2} + \frac{\partial^2 \bar{\varphi}_I}{\partial Z^2} + \frac{1}{r} \frac{\partial \bar{\varphi}_I}{\partial r} = \frac{S^2}{\alpha_i^2} \bar{\varphi}_I. \quad \dots(1.4)$$

The boundary conditions corresponding to the equation (1.4) will take the form

$$\text{at } Z = 0 - \infty < r < \infty$$

$$\rho_1 \bar{\varphi}_1 = \begin{cases} \frac{\bar{P}_0}{S + m} \\ 0 \end{cases} \quad \dots(1.5)$$

$$\text{at } Z = h - \infty < r < \infty.$$

$$\left. \begin{aligned} \rho_1 \bar{\varphi}_1 &= \rho_2 \bar{\varphi}_2 \\ \frac{\partial \bar{\varphi}_1}{\partial Z} &= \frac{\partial \bar{\varphi}_2}{\partial Z} \end{aligned} \right\} \quad \dots(1.6)$$

The solution of the equation (1.4) may be written as :

$$\bar{\varphi}_I = R(r) Z(z)$$

The general solution of equation of (1.4) will take the form

$$\bar{\varphi}_I = \int_0^\infty \left[A_I(\lambda) \exp(-\{\lambda^2 + (S^2/\alpha_i^2)\}^{1/2} Z) + B_I(\lambda) \exp(-\{\lambda^2 + (S^2/\alpha_i^2)\}^{1/2} Z) \right] J_0(\lambda r) d\lambda \quad \dots(1.7)$$

where

$A_I(\lambda)$ and $B_I(\lambda)$ are arbitrary functions obtained from the boundary conditions and J_0 is Bessel's function of zero order. Using the boundary conditions we get

$$\begin{aligned} A_1 + B_1 &= P_0 \int_0^\infty \frac{r J_0(\lambda r) dr}{S + m} \\ \rho_1 \left[A_1 \exp[-(\lambda^2 + (S^2/\alpha_1^2))^{1/2} h] + B_1 \exp[(\lambda^2 + (S^2/\alpha_1^2))^{1/2} h] \right] \\ &= \rho_2 \left[A_2 \exp[-(\lambda^2 + (S^2/\alpha_2^2))^{1/2} h] + B_2 \exp[(\lambda^2 + (S^2/\alpha_2^2))^{1/2} h] \right] \\ \sqrt{\lambda^2 + \frac{S^2}{\alpha_1^2}} \left[-A_1 \exp[-(\lambda^2 + (S^2/\alpha_1^2))^{1/2} h] + B_1 \exp[(\lambda^2 + (S^2/\alpha_1^2))^{1/2} h] \right] \\ &= \sqrt{\lambda^2 + \frac{S^2}{\alpha_2^2}} \left[-A_2 \exp[-(\lambda^2 + (S^2/\alpha_2^2))^{1/2} h] \right. \\ &\quad \left. + B_2 \exp[(\lambda^2 + (S^2/\alpha_2^2))^{1/2} h] \right] \end{aligned} \quad \dots(1.8)$$

In the case of $\alpha_1 = \alpha_2 = \alpha$, i.e. where the velocities of sound in both fluids are equal, solving the system of eqn. (1.8), we have

$$B_2 = 0$$

$$A_1 = \frac{\lambda (\rho_2 - \rho_1) P_0}{(\rho_2 + \rho_1) [1 + \exp [-2h (\lambda^2 + (S^2/\alpha^2))^{1/2}]]} \int_0^\infty \frac{r J_0 (\lambda r) dr}{S + m}$$

$$B_1 = \frac{\lambda (\rho_2 - \rho_1) P_0 \exp [-2h (\lambda^2 + (S^2/\alpha^2))^{1/2}]}{(\rho_2 + \rho_1) [1 + \exp [2h (\lambda^2 + (S^2/\alpha^2))^{1/2}]]} \int_0^\infty \frac{r J_0 (\lambda r) dr}{S + m}$$

$$A_2 = \frac{\rho_2}{\rho_1} \int_0^\infty r J_0 (\lambda r) dr.$$

Putting the values of A_1 , B_1 , A_2 and B_2 in the relation (1.7) we get

$$\begin{aligned} \bar{\varphi}_1 = & \int_0^\infty \left[\exp \left(- \sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} Z \right) + \exp (Z - 2h) \sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} \right] \\ & \times \frac{\lambda (\rho_2 - \rho_1) J_0 (\lambda r) d\lambda P_0}{(\rho_2 + \rho_1) \left[1 + \exp \left(-2h \sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} \right) \right]} \int_0^\infty \frac{r J_0 (\lambda r) dr}{S + m} \end{aligned} \quad \dots(1.9)$$

and

$$\bar{\varphi}_2 = \frac{\rho_2 P_0}{\rho_1} \int_0^\infty \exp \left(- \sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} Z \right) J_0 (\lambda r_0) \int_0^\infty \frac{r J_0 (\lambda r) dr}{S + M} \dots (1.10)$$

From the condition $\text{Re} \sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} > 0$ then we have

$$\frac{1}{1 + -2h \exp \left(\sqrt{\lambda^2 + \frac{S^2}{\alpha^2}} \right)} = \sum_{n=0}^{\infty} (-1)^n \exp [2hn (\lambda^2 + (S^2/\alpha^2))^{1/2}] \quad \dots (1.11)$$

and

$$J_0 (\lambda r) J_0 (\lambda r_0) = \frac{1}{2\pi} \int_0^\infty J_0 (\lambda R) d\theta \quad \dots(1.12)$$

where

$$R = (r^2 + r_0^2 - 2rr_0 \cos \theta)^{1/2}.$$

Putting (1.11) and (1.12) in the relations (1.9) and (1.10) we get

$$\bar{\varphi}_1 = P_0 \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \left[\frac{\exp \left(-\frac{1}{\alpha} [Z + (2n+1)h] S \right)}{S + m} \right. \\ \left. - \frac{Z + (2n+1)h}{\sqrt{S^2 + r^2}} \frac{\frac{1}{\alpha} \exp \left(\sqrt{[Z + (2n+1)h]^2 + r^2} S \right)}{S + m} \right]$$

$$\bar{\varphi}_2 = \frac{2\rho_1}{\rho_1 + \rho_2} P_0 \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \frac{\exp \left(\frac{1}{\alpha} [Z + (2n+1)h] \right)}{S + m}.$$

The corresponding solutions for φ_1 and φ_2 will take the form

$$\varphi_1 = P_0 \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \left[\exp \left(-m \left[t - \frac{Z + (2n+1)h}{\alpha} \right] \right) \right. \\ \left. - \frac{t}{\sqrt{Z + (2n+1)h - r^2}} \exp \left(-m \left[t - \frac{1}{\alpha} \sqrt{[Z + (2n+1)h]^2 + r^2} \right] \right) \right]$$

$$\varphi_2 = \frac{2\rho_1 P_0}{\rho_1 + \rho_2} \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \exp \left(m \left[t - \frac{Z + (2n+1)h}{\alpha} \right] \right).$$

According to relation

$$P - P_0 = \rho_0 \frac{\partial \varphi}{\partial t}$$

we have

$$\frac{P_1}{P_0} = \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \exp \left(-m \left[t - \frac{Z + Z + (2n+1)h}{\alpha} \right] \right) \\ - \frac{t}{\sqrt{[Z + (2n+1)h] + r^2}} \exp \left(-m \left[t - \frac{1}{\alpha} \sqrt{[Z + (2n+1)h] + r^2} \right] \right) \\ \frac{P_2}{P_0} = \frac{2\rho_1}{\rho_1 + \rho_2} \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \exp \left(m \left[t - \frac{2 + (2n+1)h}{\alpha} \right] \right).$$

In the case of incompressible fluid $\alpha = \infty$. The corresponding solution for P_1 and P_2 in this case will take the form

$$\frac{P_1}{P_0} = e^{-mt} \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n \left[1 - \frac{t}{\sqrt{[Z + (2n + 1)h]^2 + r^2}} \right]$$

$$\frac{P_2}{P_0} = \frac{2\rho_1}{\rho_1 + \rho_2} e^{mt} \sum_{n=0}^{\infty} \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right)^n.$$

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PROPAGATION CHARACTERISTICS IN DISTENSIBLE TUBES CONTAINING A DUSTY VISCOUS FLUID

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Here the propagation characteristics of a dusty viscous fluid, regarded as analogous to blood in arteries, in a distensible tube are studied. The equation of motion of the vessel wall takes into account the pulsatile nature of the wall. Results are analysed for the resistance and the reactance of the fluid and wall impedance. The influence of dust particles on these factors is noted and observation made in reference to the presence of the high haemoglobin contents in blood.

INTRODUCTION

Certain investigations have been made on blood flow in the frequency range of physiological importance^{4,7}. Kaimal² investigated the propagation characteristics of a visco-elastic fluid in a distensible tube. The present investigation is concerned with the problem of a dusty viscous fluid in distensible vessel. The vessel is modelled as a straight thin walled circular distensible tube and is externally constricted so that the longitudinal movement may be ignored. The study will be helpful in understanding the flow features of blood, especially in the coronary arteries which are subjected to varying external pressure within the muscular walls of the heart, by visualizing blood as a dusty viscous fluid exhibiting characteristics of a visco-elastic fluid as shown by Chadda¹ who compared his result with those of Tandon⁶.

MATHEMATICAL FORMULATION

The equations governing the flow of dusty viscous fluid flow in a distensible tube in cylindrical polar coordinates (r, z) with axial symmetry, following Saffman⁵ are :

$$\frac{\partial u_1}{\partial z} + \frac{u_2}{r} + \frac{\partial u_2}{\partial r} = 0 \quad \dots(1)$$

$$\frac{\partial v_1}{\partial z} + \frac{v_2}{r} + \frac{\partial v_2}{\partial r} = 0 \quad \dots(2)$$

$$\begin{aligned} \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial z} + u_2 \frac{\partial u_1}{\partial r} = & - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r} \frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial z^2} \right) \\ & + \frac{k N}{\rho} (v_1 - u_1) \end{aligned} \quad \dots(3)$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_2}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r} \frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{\partial^2 u_2}{\partial z^2} \right) + \frac{kN}{\rho} (v_2 - u_2) \quad \dots(4)$$

$$m \left\{ \frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial z} + v_2 \frac{\partial v_1}{\partial r} \right\} = k (u_1 - v_1) \quad \dots(5)$$

and

$$m \left\{ \frac{\partial v_2}{\partial t} + u_1 \frac{\partial v_2}{\partial z} + v_2 \frac{\partial v_2}{\partial r} \right\} = k (u_2 - v_2) \quad \dots(6)$$

where (u_1, v_1) and (u_2, v_2) are respectively, the velocities of the fluid and dust particles, p is fluid pressure ρ the density of the fluid, m the mass of a dust particle, k the Stokes' resistance coefficient, N the number density of dust particles, assumed to be constant, and ν the kinematic coefficient of viscosity.

Ignoring the longitudinal components, the motion of the thin walled tube, under the initial conditions $\xi(0) = \xi_0$, $\dot{\xi}(0) = \dot{\xi}_0$ is described by Knowles³ as

$$\frac{d^2 \xi}{dt^2} + \left\{ \bar{K} - \frac{(\bar{p}_1 - \bar{p}_2)}{\frac{1}{2} \mu \bar{\rho}_m} \right\} \xi - \bar{K} \xi^{-3} = 0 \quad \dots(7)$$

where

$$\bar{K} = \frac{K r_1^4}{\nu^2}, \quad K = \frac{\alpha + \beta}{\rho_m r_1^2}, \quad \bar{\rho}_m = \frac{\rho_m}{\rho}, \quad \bar{t} = \frac{t \nu}{r_1^2}.$$

α, β are material constants and ρ_m is density of the material, r_1 is the internal radius of the tube and quantities with bars denote corresponding non-dimensional quantities. For the quantity $(\bar{p}_1 - \bar{p}_2)$, being the difference between pressure on two sides of the tube wall, we assume that

$$\frac{\bar{p}_1 - \bar{p}_2}{\frac{1}{2} \mu \bar{\rho}_m} = A(\bar{z}) \psi(\bar{\omega} \bar{t}) \quad \dots(8)$$

where $\psi(\bar{\omega} \bar{t})$ is a periodic function, $|\psi(\bar{\omega} \bar{t})| < 1$ and $\bar{z} = \frac{z}{r_1}$. The homogeneous part of eqn. (7), using eqn. (8) is

$$\frac{d^2 \xi}{d\bar{t}^2} + \{ \bar{K} - A \psi(\bar{\omega} \bar{t}) \} \xi - \bar{K} \xi^{-3} = 0 \quad \dots(9)$$

which is a generalized Hills equation.

SOLUTION OF THE PROBLEM

The independent solutions of equation (9) are

$$\xi_1(\bar{t}) = C_1 \cos \left(\bar{K}^{1/2} \int_0^{\bar{t}} \phi(\bar{t}) d\bar{t} \right) \quad \dots(10)$$

and

$$\xi_2(\bar{t}) = C_2 \sin(\bar{K}^{1/2} \int_0^{\bar{t}} \phi(\bar{t}) d\bar{t}) \quad \dots(11)$$

where

$$\left. \begin{aligned} \phi(\bar{t}) &= \sigma - (2\sigma\ddot{\sigma} - 3\dot{\sigma}^2) / 8\bar{K}\sigma^3 \\ \sigma^2 &= 1 - \bar{\alpha}\psi(\bar{\omega}\bar{t}), \bar{\alpha} = A/\bar{K} \\ C_1 &= \xi_0 \sigma(0)^{1/2} \end{aligned} \right\} \quad \dots(12)$$

and

$$C_2 = \bar{K}^{-1/2} \left\{ \xi'_0 \sigma(0)^{1/2} + \frac{1}{2} \xi_0 \dot{\sigma}(0)^{3/2} \right\} / \phi(0).$$

It may be noticed that

$$\left. \begin{aligned} \xi_1(0) &= \xi_0, \dot{\xi}_1(0) = \xi'_0 \\ \xi_2(0) &= \xi_0, \dot{\xi}_2(0) \neq 0. \end{aligned} \right\} \quad \dots(13)$$

Therefore, the solution of the non linear equation (7) using initial conditions is

$$\xi(t) = \frac{[(\bar{A}^4 \phi^2 + \sigma^2) + (\bar{A}^4 \phi^2 - \sigma^2) \cos(2\bar{K}^{1/2} \int_0^{\bar{t}} \phi(\bar{t}) d\bar{t})]}{\sqrt{2} \xi_0 \sigma(0)^{1/2} \phi \sigma^{1/2}} \quad \dots(14)$$

This solution is stable and real only if $\bar{\alpha} < 1$ and $\bar{\alpha}$ signifies the influence of fluid on the motion of wall of the tube.

As a more specific case, let $\psi(\bar{\omega}\bar{t}) = \sin \bar{\omega}\bar{t}$, $\xi_0 = 1$, $\xi'_0 \neq 0$ so that

$$\sigma(0) = 1, C_1 = 1. \quad \dots(15)$$

If we assume that such a dusty viscous fluid may be regarded as an approximation to blood, then in most physiological situations ω^2/\bar{K} and higher powers may be neglected. Therefore function $\phi(\bar{t})$ can be approximately taken as $\sigma(\bar{t})$, consequent to which eqn. (14) with eqn. (15) reduces to

$$\xi(\bar{t}) = \frac{1}{\sigma^{1/2}} = (1 - \bar{\alpha} \sin \bar{\omega}\bar{t})^{-1/4} \quad \dots(16)$$

which suggests that the general solution of the radial component of oscillatory motion of the tube may be given by

$$\xi(t) = \{1 - \bar{\alpha} \exp i(\bar{\omega}\bar{t})\}^{-1/4} \quad \dots(17)$$

Here the motion of the wall is determined by the pressure difference between the two sides of the wall. Equation (17), when modified to include an exponentially decaying factor in the axial direction, takes the form

$$\xi(\bar{z}, \bar{t}) = \{1 - \bar{\alpha} \exp i(\bar{\omega} \bar{t}) \bar{\xi}^{1/4} - \bar{e} i \bar{k}_0 \bar{z}\} \quad \dots(18)$$

where \bar{k}_0 is the dimensionless wave number. To solve highly nonlinear equations (3)-(6), equations are linearized by introducing long wave-length approximations and are transformed by

introducing non-dimensional quantities :

$$\begin{aligned} \bar{u} &= \frac{u_1 r_1}{v}, \bar{u}_2 = \frac{u_2 r_1}{v}, \bar{v}_1 = \frac{v_1 r_1}{v}, \bar{v}_2 = \frac{v_2 r_1}{v}, \bar{t} = \frac{t v}{r_1^2} \\ \tau &= \frac{m v}{k r_1^2}, f = \frac{m N}{\rho}, \bar{r}_1 = \frac{r}{r_1}, \bar{z} = \frac{z}{r_1}, \bar{p} = \frac{p r_1^2}{\rho v^2}, \bar{\omega} = \frac{\omega r_1^2}{v}. \end{aligned}$$

Hence, eqns. (3) — (6) are reduced to

$$\frac{\partial \bar{u}_1}{\partial \bar{t}} + \frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial^2 \bar{u}_1}{\partial \bar{r}_2^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}_1}{\partial \bar{r}} + \frac{f}{\tau} (\bar{v}_1 - \bar{u}_1) \quad \dots(19)$$

$$\frac{\partial \bar{u}_2}{\partial \bar{t}} + \frac{\partial \bar{p}}{\partial \bar{r}} = \frac{\partial^2 \bar{u}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}_2}{\partial \bar{r}} + \frac{f}{\tau} (\bar{v}_2 - \bar{u}_2) \quad \dots(20)$$

$$\frac{\partial \bar{v}_1}{\partial \bar{t}} = \frac{1}{\tau} (\bar{u}_1 - \bar{v}_1) \quad \dots(21)$$

and

$$\frac{\partial \bar{u}_2}{\partial \bar{t}} = \frac{1}{\tau} (\bar{u}_2 - \bar{v}_2) \quad \dots(22)$$

where f is mass concentration of the dust particles and τ is the relaxation time, with boundary conditions :

$$\bar{u} = 0, \bar{u}_2 = \bar{\xi}(\bar{t}) \text{ at } r = 1. \quad \dots(23)$$

Using the boundary conditions, the velocity components of the fluid and the dust particles are given by

$$\bar{v}_1 = \frac{\bar{u}_1}{1 + \tau \bar{\omega} + \frac{5/4 \tau i \bar{\omega} \bar{\alpha} \exp i(\bar{\omega} \bar{t})}{1 - \bar{\alpha} \exp i(\bar{\omega} \bar{t})}} \quad \dots(24)$$

$$\bar{v}_2 = \frac{\bar{u}_2}{1 + \tau_i \bar{\omega} + \frac{5/4 \tau_i \bar{\omega} \bar{\alpha} \exp i (\bar{\omega} \bar{t})}{1 - \bar{\alpha} \exp i (\bar{\omega} \bar{t})}} \quad \dots(25)$$

$$\bar{u}_1 = \frac{p_0}{(\delta^2 - n^2)} \left\{ \frac{n J_0(n) J_0(\delta \bar{r})}{J_0(\delta)} - n J_0(n \bar{r}) \right\} \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z}) \quad \dots(26)$$

$$\bar{u}_2 = \left[\left\{ A_1 + \frac{n p_0}{(\delta^2 - n^2)} J_1(n) \right\} \frac{J_1(\delta \bar{r})}{J_1(\delta)} - \frac{n p_0}{(\delta^2 - n^2)} J_1(n \bar{r}) \right] \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z}) \quad \dots(27)$$

where

$$\delta^2 = \frac{f}{\tau} \left\{ \frac{1}{1 + \tau_i \bar{\omega} + \frac{5 \tau_i \bar{\omega} \bar{\alpha} \exp i (\bar{\omega} \bar{t})}{4 (1 - \bar{\alpha} \exp i (\bar{\omega} \bar{t}))}} - 1 - \frac{\tau_i \bar{\omega}}{f} - \frac{5 \tau_i \bar{\omega} \bar{\alpha} \exp i (\bar{\omega} \bar{t})}{4 f (1 - \bar{\alpha} \exp i (\bar{\omega} \bar{t}))} \right\}$$

and

$$\bar{p} = p_0 J_0(n \bar{r}) \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z})$$

$$n = i \bar{k}_0. \quad \dots(28)$$

Following Womersley⁸, the components of velocity of the fluid u_1, u_2 and those of the dust particles v_1, v_2 , for $n^2 \ll \delta^2$ are given by

$$\bar{u} = \frac{n p_0}{\delta^2} J_0(n) \left\{ \frac{J_0(\delta \bar{r})}{J_0(\delta)} - \frac{J_0(n \bar{r})}{J_0(n)} \right\} \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z}) \quad \dots(29)$$

$$\bar{u}_2 = \left[\left\{ A_1 + \frac{n p_0}{\delta^2} J_1(n) \right\} \frac{J_1(\delta \bar{r})}{J_1(\delta)} - \frac{n p_0}{\delta^2} J_1(n \bar{r}) \right] \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z}) \quad \dots(30)$$

$$\bar{v}_1 = \frac{\bar{u}_1}{1 + \tau_i \bar{\omega} + \frac{5/4 \tau_i \bar{\omega} \bar{\alpha} \exp i (\bar{\omega} \bar{t})}{1 - \bar{\alpha} \exp i (\bar{\omega} \bar{t})}} \quad \dots(31)$$

$$\bar{v}_2 = \frac{\bar{u}_2}{1 + \tau_i \bar{\omega} + \frac{5/4 \tau_i \bar{\omega} \bar{\alpha} \exp i (\bar{\omega} \bar{t})}{1 - \bar{\alpha} \exp i (\bar{\omega} \bar{t})}} \quad \dots(32)$$

The rate of flow Q in the lumen is given by

(taking $J_0(n) = 1, J_1(n) = n/2$)

$$Q = \frac{\pi n p_0}{\delta^2} \exp i (\bar{\omega} \bar{t} - \bar{k}_0 \bar{z}) \left\{ \frac{2 J_1(\delta)}{\delta J_0(\delta)} - 1 \right\}. \quad \dots(33)$$

The average pressure over the cross section of the tube is given by

$$p(\bar{z}, \bar{t}) = \frac{2p_0}{n} J_1(n) \exp i(\bar{\omega}\bar{t} - \bar{k}_0 \bar{z}). \quad \dots(34)$$

The longitudinal impedance Z is given by

$$Z = - \frac{\partial^2}{\pi} \left\{ \frac{2}{1 - \frac{2J_1(\delta)}{\delta J_0(\delta)}} \right\} \quad \text{(approximately).} \quad \dots(35)$$

The limiting value of Z for small value of δ is given by

$$Z_s = - \frac{1}{\pi} \{ 2\delta^2 - 8 \}. \quad \dots(36)$$

Here in the limiting case, fluid resistance R_s and fluid reactance $(\omega L)_s$, are given by

$$R_s = - \frac{2}{\pi} \left[\frac{f}{\tau} \left\{ \frac{\tau^2 \bar{\omega}^2}{1 + \tau^2 \bar{\omega}^2} - \frac{5\tau \bar{\omega} \bar{\alpha}}{4(1 + \tau^2 \bar{\omega}^2)^2} (2\tau \bar{\omega} \cos \bar{\omega} \bar{t} - \sin \bar{\omega} \bar{t} + \tau^2 \bar{\omega} \sin \bar{\omega} \bar{t} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4f} \sin \bar{\omega} \bar{t}) \right\} - 4 \right] \quad \dots(37)$$

and

$$(\omega L)_s = \frac{2}{\pi} \left[\frac{f}{\tau} \left\{ \frac{\tau \bar{\omega}}{1 + \tau^2 \bar{\omega}^2} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4(1 + \tau^2 \bar{\omega}^2)} (\cos \bar{\omega} \bar{t} - \tau^2 \bar{\omega}^2 \cos \bar{\omega} \bar{t} + 2\tau \bar{\omega} \sin \bar{\omega} \bar{t} + \frac{\tau \bar{\omega}}{f} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4f} \cos \bar{\omega} \bar{t}) \right\} \right]. \quad \dots(38)$$

The polar form of Z_s (for small values of δ) is also given by

$$\begin{aligned} |Z_s| = & \frac{2f}{\pi\tau} \left[\left\{ \frac{\tau^2 \bar{\omega}^2}{1 + \tau^2 \bar{\omega}^2} + \frac{4\tau}{f} \right\}^2 + \tau^2 \bar{\omega}^2 \left\{ \frac{1}{1 + \tau^2 \bar{\omega}^2} + \frac{1}{f} \right\}^2 \right. \\ & + \frac{5}{2} \tau \bar{\omega} \bar{\alpha} \left\{ \left(\frac{\tau^2 \bar{\omega}^2}{1 + \tau^2 \bar{\omega}^2} + \frac{4\tau}{f} \right) \right. \\ & \times \left(\frac{2\tau \bar{\omega} \cos \bar{\omega} \bar{t} - (1 - \tau^2 \bar{\omega}^2) \sin \bar{\omega} \bar{t}}{(1 + \tau^2 \bar{\omega}^2)^2} \right. \\ & \left. \left. - \frac{\sin \bar{\omega} \bar{t}}{f} \right) + \tau \bar{\omega} \left(\frac{1}{(1 + \tau^2 \bar{\omega}^2)} + \frac{1}{f} \right) \right] \end{aligned}$$

(equation continued on p. 912)

$$\times \left(\frac{(1 - \tau^2 \bar{\omega}^2) \cos \bar{\omega} \bar{t} + 2\tau \bar{\omega} \sin \bar{\omega} \bar{t}}{(1 + \tau^2 \bar{\omega}^2)^2} + \frac{\cos \bar{\omega} \bar{t}}{f} \right) \Bigg]^{1/2} \quad \dots(39)$$

and

$$\psi_s = \frac{\left[\frac{f}{\tau} \left\{ \frac{\tau \bar{\omega}}{1 + \tau^2 \bar{\omega}^2} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4(1 + \tau^2 \bar{\omega}^2)^2} (\cos \bar{\omega} \bar{t} - \tau^2 \bar{\omega}^2 \cos \bar{\omega} \bar{t} + 2\tau \bar{\omega} \sin \bar{\omega} \bar{t}) + \frac{\tau \bar{\omega}}{f} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4f} \cos \bar{\omega} \bar{t} \right\} \right]}{\left[\frac{f}{f} \left\{ \frac{\tau^2 \bar{\omega}^2}{1 + \tau^2 \bar{\omega}^2} + \frac{5\tau \bar{\omega} \bar{\alpha}}{4(1 + \tau^2 \bar{\omega}^2)^2} (2\tau \bar{\omega} \cos \bar{\omega} \bar{t} - \sin \bar{\omega} \bar{t} + \tau^2 \bar{\omega}^2 \sin \bar{\omega} \bar{t}) - \frac{5\tau \bar{\omega} \bar{\alpha}}{4f} \sin \bar{\omega} \bar{t} \right\} + 4 \right]}. \quad \dots(40)$$

DISCUSSION

Noting that parameter τ is a measure of the time for the dust particles to adjust to changes in the fluid velocity, it is called relaxation time and is proportional to the size of the individual particles. Also f signifies the amount of dust particles present. From the results obtained above we can discuss the influence of dust particles on wave propagation characteristics and on the pressure flow relationship in the tube. As the tube wall is subjected to external pressure the results can be applied to study the flow

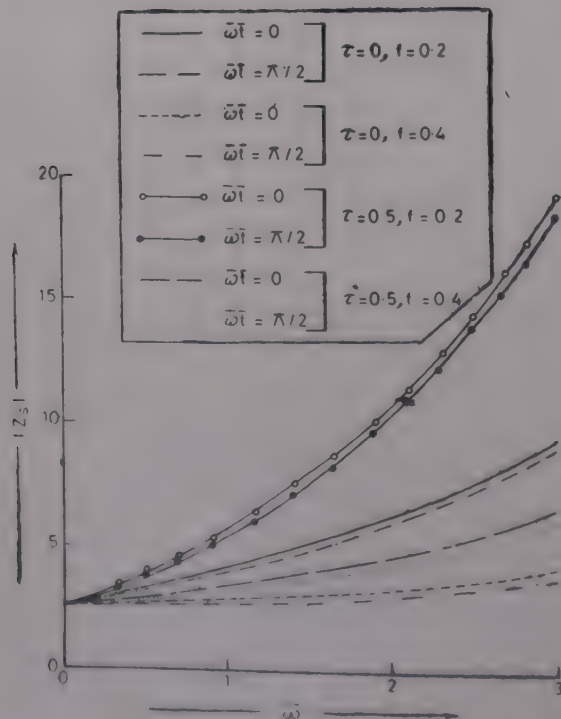


FIG. 1. Magnitude $|Z_s|$ of the wall
Impedance Against $\bar{\omega}$.

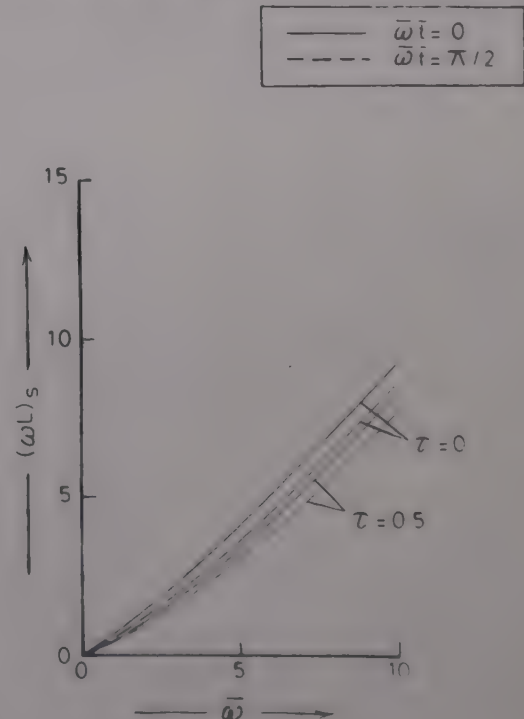
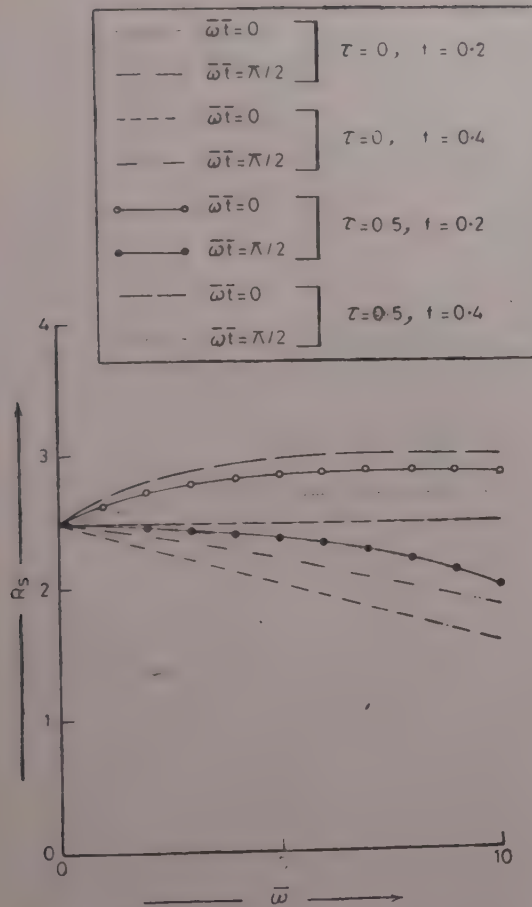
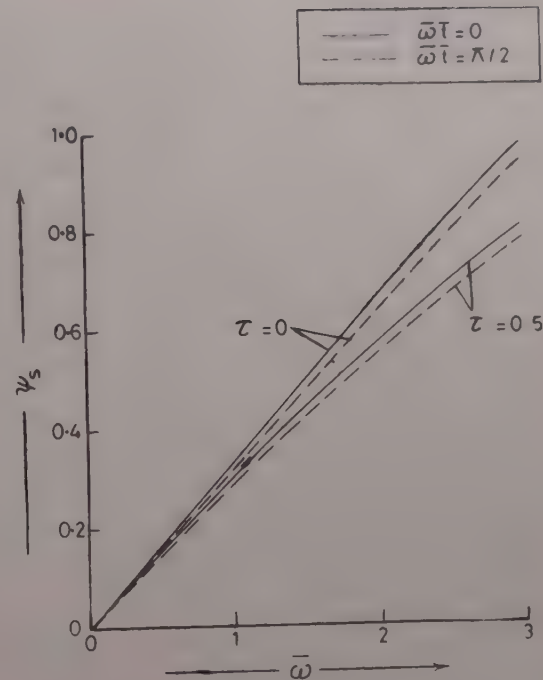


FIG. 2. The impedance angle ψ_s
against $\bar{\omega}$.

FIG. 3. The fluid Resistance R_s Against $\bar{\omega}$.FIG. 4. The fluid Reactance $(\omega L)_s$ against $\bar{\omega}$.

in the coronary arteries where much of their length is imbedded in the muscular walls of the heart and are subjected to varying external pressure. Equation (35) gives the longitudinal impedance due to the flow and equation (36) gives the impedance for small values of δ . The behaviour of transverse wall impedance is plotted in Figs. 1 and 2 against various values of $\bar{\omega}$ when the value of $\bar{\alpha}$ is taken to be 0.1 in all cases. These graphs depict the effects of wall oscillation and presence of dust particles in fluid resistance and reactance respectively. It is seen that effects of dust particles and wall pulsation are more significant for higher value of $\bar{\omega}$.

Figure 1 shows that the increase in the value of f from 0.2 to 0.4 causes decrease in the impedance. This has the effect of bringing the frequency of oscillation of tube closer to its natural frequency, while Fig. 3 depicts increase in resistance with the increase in the value of f . Since increase in f is due to the increase in the concentration of the dust particles. We see that if there is increase in the cell contents of blood in arteries it will cause increase in the resistance.

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DISTURBANCE IN A NON-HOMOGENEOUS ELASTIC MEDIUM BY A TWISTING IMPULSIVE FORCE

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The propagation of disturbance in a non-homogeneous elastic medium due to a twisting impulsive force acting on the surface of a spherical cavity has been discussed. The shear modulus of the material has been assumed to be proportional to an arbitrary, not necessarily integral, power of the radial distance from the centre of the cavity, while no restriction has been placed upon the continuous radial variation of Young's modulus. In a particular case the propagation of disturbance and stresses have been shown graphically at different positions and at different times. The disturbance and stresses in the associated homogeneous medium has been shown in details.

1. INTRODUCTION

The propagation of shock waves in an isotropic homogeneous elastic plate of infinite extent and arbitrary thickness under an impulsive twist applied on the boundary of the transverse cylindrical hole was discussed by Goodier and Jahsman¹. Sternberg and Chakravorty² extended that discussion for an elastic medium of non-homogeneous elastic moduli. Dutta³ obtained the solution of the problem in which the disturbance is generated in a non-homogeneous isotropic medium due to a twisting impulsive force applied on the surface of a spherical cavity. In his paper Dutta³ assumed such non-homogeneity of the medium that the governing differential equation could easily be solved. In the present paper we consider the same problem but with different and more general type of non-homogeneity arising out of the variation of shear modulus with the radial distance having an arbitrary exponent. If μ denotes the shear modulus of the material, we assume

$$\mu = \mu_0 \left(\frac{r}{a} \right)^\alpha, \quad \left(\alpha \begin{matrix} \geq \\ < \end{matrix} 0 \right) \quad \dots (1)$$

in which μ_0 is the value of μ on the surface of the cavity, a being its radius.

In dealing with their problem in homogeneous medium, Goodier and Jahsman¹ made use of the operational scheme originated by Kromm⁴ in connection with an allied problem. In the present paper, if $\alpha \neq 2$, Kromm's procedure has been extended to the present generalised problem. We are thus led to the solutions of integral equations for the determination of displacement and stresses.

For the exceptional case $\alpha = 2$, we deduce an exact explicit solution in integral forms.

Finally we find that there exists an infinite sequence of α -values which give rise to exact solutions in closed form and in terms of elementary functions. One member of this sequence, which corresponds to $\alpha = -4/3$ is determined explicitly and discussed numerically. To illustrate the effect of non-homogeneity the case $\alpha = 0$ (homogeneous shear modulus) has been exhibited with figures.

It should be emphasized that the admission of a power law for the density-analogous to that stipulated for μ , but involving an exponent which may be different from α , introduces no essential complication.

2. FORMULATION OF THE PROBLEM

Since the problem under study is characterised by a displacement due to a twisting impulsive pressure, we choose the components of displacement u_r, u_θ, u_ϕ in spherical polar coordinates (r, θ, ϕ) as

$$u_r = u_\theta = 0 \text{ and } u_\phi = u(r, t) \quad \dots(2)$$

where t denotes time.

The only non-vanishing stress component is given by

$$\widehat{r\phi} = S(r, t) = \mu \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right). \quad \dots(3)$$

Two of the equations of motion are identically satisfied and the third reduces to

$$\frac{\partial S}{\partial r} + \frac{3S}{r} = \gamma \cdot \frac{\partial^2 u}{\partial t^2} \quad \dots(4)$$

in which γ designates the constant mass density.

Assuming the shear modulus μ to be an arbitrary function of r , differentiable in $a \leq r < \infty$, (4) in view of (3) becomes

$$\mu \frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial \mu}{\partial r} + \frac{2\mu}{r} \right) \frac{\partial u}{\partial r} - \left(\frac{1}{r} \frac{\partial \mu}{\partial r} + \frac{2\mu}{r^2} \right) u = \gamma \frac{\partial^2 u}{\partial t^2}. \quad \dots(5)$$

The relevant initial conditions are

$$u(r, 0) = 0, \quad \left[\frac{\partial u}{\partial t} \right]_{(r,0)} = 0 \quad (a < r < \infty) \quad \dots(6)$$

while the boundary condition is

$$S(a, t) = -S_0 \delta(t) \quad (-\infty < t < \infty) \quad \dots(7)$$

in which $\delta(t)$ is Dirac's delta function as defined by

$\delta(t)$ is very large in a vanishingly small region to the right of $t = 0$ and is zero elsewhere :

$$\int_{-\infty}^{\infty} \delta(t) dt = 1; \quad \dots(8)$$

Since the tractions at infinite distance are to vanish, we adjoin the regularity condition.

$$S(r, t) \rightarrow 0 \text{ as } r \rightarrow \infty. \quad \dots(9)$$

We now define the dimensionless radial coordinate and time by means of the relations

$$\rho = \frac{r}{a}, \tau = \frac{c_0 t}{a} \quad \dots(10)$$

where

$$c_0 = (\mu_0/\gamma)^{1/2}, \mu_0 = \mu \text{ at } (a)$$

whence c_0 is the velocity of the shear waves in a homogeneous medium whose shear modulus coincides with the local value of μ at the boundary of the cavity. Next we define a dimensionless shear modulus, displacement and stress through

$$\bar{\mu} = \frac{\mu}{\mu_0}, \bar{u} = \frac{\mu_0 u}{S_0 a}, \bar{S} = \frac{S}{S_0}. \quad \dots(11)$$

Equations (3) and (5) with the aid of (10) and (11) become

$$\bar{S} = \bar{\mu} \left[\frac{\partial \bar{u}}{\partial \rho} - \frac{\bar{u}}{\rho} \right] \quad \dots(12)$$

$$\frac{\partial^2 \bar{u}}{\partial \rho^2} + \left(2 + \frac{\rho}{\mu} \frac{d\bar{\mu}}{d\rho} \right) \left(\frac{1}{\rho} \frac{\partial \bar{u}}{\partial \rho} - \frac{\bar{u}}{\rho^2} \right) = \frac{1}{\bar{\mu}} \frac{\partial^2 \bar{u}}{\partial \tau^2}. \quad \dots(13)$$

The partial differential equation (13) is analytically tractable if

$$\bar{\mu} = \rho^\alpha, \left(\alpha \begin{matrix} > \\ < \end{matrix} 0 \right). \quad \dots(14)$$

We shall assume (14) and consequently replace (12) and (13) with

$$\bar{S} = \rho^\alpha \left[\frac{\partial \bar{u}}{\partial \rho} - \frac{\bar{u}}{\rho} \right] \quad \dots(15)$$

$$\frac{\partial^2 \bar{u}}{\partial \rho^2} + (2 + \alpha) \left(\frac{1}{\rho} \frac{\partial \bar{u}}{\partial \rho} - \frac{\bar{u}}{\rho^2} \right) = \frac{1}{\rho^\alpha} \frac{\partial^2 \bar{u}}{\partial \tau^2}. \quad \dots(16)$$

The conditions (6), (7) and (9) by virtue of (10) and (11) may be written as

$$\bar{u}(\rho, 0) = 0, \left[\frac{\partial \bar{u}}{\partial \tau} \right]_{(\rho, 0)} = 0, (1 < \rho < \infty) \quad \dots(17)$$

$$S(1, \tau) = -\delta(\tau), (-\infty < \tau < \infty) \quad \dots(18)$$

$$S(\rho, \tau) \rightarrow 0 \text{ as } \rho \rightarrow \infty. \quad \dots(19)$$

Thus we need to determine the solution of (16) which conforms to (17) and is such that the stress (15) meets (18) and (19).

3. GENERAL SOLUTION

Let the Laplace transform of $f(\rho, \tau)$ with respect to τ be $F(\rho, p)$ such that

$$F(\rho, p) = L[f(\rho, \tau)] = \int_0^\infty e^{-p\tau} f(\rho, \tau) d\tau \quad \dots(20)$$

in which p is the transform parameter.

In particular let

$$U(\rho, p) = L[\bar{u}(\rho, \tau)]$$

$$S(\rho, p) = L[\bar{S}(\rho, \tau)]. \quad \dots(21)$$

Applying Laplace transform to (15), (16), (18) and (19) and taking into account the initial condition (17) we get the field equations

$$S = \rho^\alpha \left[\frac{dU}{dp} - \frac{U}{p} \right] \quad \dots(22)$$

$$\frac{d^2 U}{d\rho^2} + \frac{2+\alpha}{\rho} \frac{dU}{d\rho} - \left[\frac{2+\alpha}{\rho^2} + \frac{p^2}{\rho^\alpha} \right] U = 0 \quad \dots(23)$$

together with the boundary condition

$$S(1, p) = -1 \quad \dots(24)$$

and the regularity condition

$$S(\rho, p) \rightarrow 0 \text{ as } \rho \rightarrow \infty. \quad \dots(25)$$

If $\alpha \neq 2$, (23) is reducible to a modified Bessel equation and has the general solution⁵

$$U(\rho, p) = A_1 \rho^{-(\alpha+1)/2} I_n \left(\frac{p \rho^\beta}{\beta} \right) + A_2 \rho^{-(\alpha+1)/2} K_n \left(\frac{p \rho^\beta}{\beta} \right) \quad \dots(26)$$

where I_n and K_n are the modified Bessel functions of the first and second kind of order n , while

$$\beta = 1 - \frac{\alpha}{2}, n = \frac{3+\alpha}{2-\alpha}. \quad \dots(27)$$

The coefficients A_1 and A_2 appearing in (26) are arbitrary functions of p and are to be determined consistent with (24) and (25).

On the other hand for $\alpha = 2$, ($\beta = 0$) (23) is an equation of the Euler type, whose complete solution is given by

$$U = \beta_1 \rho^{m_1} + \beta_2 \rho^{m_2}$$

$$(m_1, m_2) = \frac{1}{2} (-3 \pm \sqrt{4p^2 + 25}). \quad \dots(28)$$

The character of the solution (26) depends on whether $\alpha < 2$ or $\alpha > 2$. Thus we distinguish and deal separately with following three cases :

Case I : $-\infty < \alpha < 2$, ($0 < \beta < \infty$, $-1 < n < \infty$);

Case II : $\alpha = 2$, ($\beta = 0$);

Case III : $2 < \alpha < \infty$, ($-\infty < \beta < 0$, $-\infty < n < -1$).

4. SOLUTION FOR CASE I

In this case $\beta > 0$ and the argument of the Bessel function is positive for $p > 0$ and tends to infinity as $\rho \rightarrow \infty$.

Applying conditions (24) and (25) we get from (26) and (22) with the help of the recurrence relations appropriate to Bessel functions⁶

$$U(\rho, p) = K_n \left(\frac{p \rho^\beta}{\beta} \right) [p \rho^{(\alpha+1)/2} K_{n+1}(p/\beta)]^{-1} \quad \dots(29)$$

$$S(\rho, p) = -K_{n+1} \left(\frac{p \rho^\beta}{\beta} \right) [\rho^{1/2} K_{n+1}(p/\beta)]^{-1}. \quad \dots(30)$$

As the inverse Laplace transform of the right hand side of (29) and (30) cannot be obtained from, the table of transforms, we extend Kromm's scheme for the solution in the following manner.

Schlafli's representation of $K_n(z)$ is⁷,

$$K_n(z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} (z/2)^n \int_1^\infty e^{-zx} (x^2 - 1)^{n-1/2} dx. \quad \dots(31)$$

Putting $z = \frac{p \rho^\beta}{\beta}$ and changing the variable of integration to

$$\xi = \frac{\rho^\beta x - 1}{\beta}$$

(31) becomes

$$\frac{2^{n-1} (2n-1) \Gamma(n - \frac{1}{2}) \beta^{n-1} \rho^{\beta n}}{\Gamma(\frac{1}{2}) p^n} e^{p/\beta} K_n \left(\frac{p \rho^\beta}{\beta} \right)$$

$$= \int_0^\infty h(\xi - \eta) e^{-p\xi} [(\beta\xi + 1)^2 - \rho^{2\beta}]^{n-1/2} d\xi \quad \dots(32)$$

where $h(t)$ is the Heaviside step function, defined by

$$h(t) = 1 \text{ for } t > 0, h(t) = 0 \text{ for } t < 0$$

and

$$\eta = \frac{\rho^\beta - 1}{\beta} \quad \dots(33)$$

Similarly putting $z = p/\beta$ and changing the variable of integration to

$$\xi = \frac{x-1}{\beta}$$

we get

$$\begin{aligned} & \frac{2^{n-1} \Gamma(n - \frac{1}{2}) \beta^{n-2}}{\Gamma(\frac{1}{2}) p^{n-1}} e^{p/\beta} K_{n-1}(p/\beta) \\ &= \int_0^\infty e^{-p\xi} [(\beta\xi + 1)^2 - 1]^{n-3/2} d\xi. \end{aligned} \quad \dots(34)$$

Now taking recourse to recurrence relations of modified Bessel function we rewrite as⁶

$$\begin{aligned} & \rho^{\alpha+1/2} U(\rho, p) \left[\left\{ \frac{2^{n-1} (2n-1) \Gamma(n - \frac{1}{2}) \beta^{n-1}}{\Gamma(\frac{1}{2}) p^n} e^{p/\beta} K_n(p/\beta) \right\} \right. \\ & \quad \times \left. \frac{2n}{2n-1} + \frac{2^{n-1} \Gamma(n - \frac{1}{2}) \beta^{n-2}}{\Gamma(\frac{1}{2}) p^{n-1}} e^{p/\beta} K_{n-1}(p/\beta) \right] \\ &= \left[\frac{2^{n-1} (2n-1) \Gamma(n - \frac{1}{2}) \beta^{n-1} \rho^{\beta n}}{\Gamma(\frac{1}{2}) p^n} e^{p/\beta} K_n\left(\frac{p\rho^\beta}{\beta}\right) \right] \frac{1}{\beta p^{\beta n} (2n-1)}. \end{aligned}$$

By the convolution theorem⁸ we get from (32), (34) and (35) an integral equation

$$\int_0^\tau \bar{u}(\rho, \xi) K(\tau - \xi) d\xi \quad \dots(35)$$

$$= \frac{h(\tau - \eta)}{\beta \rho^{\alpha+2}} [(\beta\tau + 1)^2 - \rho^{2\beta}]^{n-1/2} \quad \dots(36)$$

where

$$K(\xi) = [(\beta\xi + 1)^2 - 1]^{n-3/2} [2n(\beta\xi + 1)^2 - 1]. \quad \dots(37)$$

To find the stress we rewrite (30) as

$$\begin{aligned} & S(\rho, p) \left[\frac{2^{n+1} \Gamma(n + \frac{3}{2}) \beta^{n-2}}{\Gamma(\frac{1}{2}) p^{n-1} (2n+1)} e^{p/\beta} K_{n+1}(p/\beta) \right] \\ &= -p^{-3} \left[\frac{2^{n+1} \Gamma(n + \frac{3}{2}) \beta^{n-1} \rho^{\beta n + \beta}}{(2n+1) \Gamma(\frac{1}{2}) p^{n-1}} e^{p/\beta} K_{n+1}\left(\frac{p\rho^\beta}{\beta}\right) \right] \end{aligned} \quad (38)$$

Replacing n by $n+1$ in (31), changing the variable of integration to ξ and integrating by parts we get

$$\frac{2^{n+1} \Gamma(n + \frac{\alpha}{2}) \beta^{n-2} \rho^{\beta n + \beta}}{(2n + 1) \Gamma(\frac{1}{2}) p^{n-1}} e^{p/\beta} K_{n+1} \left(\frac{p \rho^\beta}{\beta} \right) \\ = \int_0^\infty h(\xi - \eta) e^{-p\xi} \phi(\rho, \xi) d\xi \quad \dots(39)$$

where

$$\phi(\rho, \xi) = [(\beta\xi + 1)^2 - \rho^{2\beta}]^{n-3/2} [2n(\beta\xi + 1)^2 - \rho^{2\beta}]. \quad \dots(40)$$

By the convolution theorem we get an integral equation of $S(\rho, \tau)$ given by

$$\int_0^\tau \bar{S}(\rho, \tau) K(\tau - \xi) d\xi = -\rho^{-3} h(\tau - \eta) \phi(\rho, \tau) \quad \dots(41)$$

where $k(\xi)$ and $\phi(\rho, \xi)$ are given by (37) and (40) respectively.

Equations (36) and (41) complete the solution of the problem. The solution of (36) and (41), in analytical forms, become feasible only for an exceptional sequence of α -values which will be given later on. For arbitrary α ($-\infty < \alpha < 2$) the values of $\bar{u}(\rho, \tau)$ and $\bar{S}(\rho, \tau)$ may be found by Kromm's method. Thus each integral equation is replaced with an (approximately) equivalent finite system of linear algebraic equations, obtained by regarding the unknown function as piece-wise constant over the range of integration. The integrals (36) and (41) are thereby reduced to integration of $k(\xi)$, called the kernel function, given by (37). Furthermore the integral of $k(\xi)$ is found to be elementary.

5. SOLUTION FOR CASE II

In the present case $\alpha = 2$ and $\beta = 0$. The general solution of the transformed displacement equation of motion is now given by (28) which on invoking (22), (24) and (25) yields

$$U(\rho, p) = \frac{2 \exp[-\frac{1}{2} \lambda (4p^2 + 25)^{1/2}]}{\rho^{3/2} [5 + (4p^2 + 25)^{1/2}]} \quad \dots(42)$$

where

$$\lambda = \log \rho. \quad \dots(43)$$

Finally the inversion formula of Laplace transforms gives, Erdelyi⁹

$$\bar{u}(\rho, \tau) = \frac{1}{\rho^{3/2}} [f(\tau) - \frac{\lambda}{2} \int_0^\tau f \sqrt{(\tau^2 - y^2)} J_1(\frac{\lambda}{2} y) dy] \quad \dots(44)$$

where

$$f(t) = h(t - \lambda) e^{-5(t-\lambda)/2} \quad \dots(45)$$

and λ is given by (43).

Also

$$S(\rho, p) = -\frac{1}{\sqrt{\rho}} \exp(-\lambda \sqrt{p^2 + \frac{15}{4}}) \quad \dots(46)$$

Hence by Carslaw and Jaeger¹⁰ and Erdelyi² we get by inversion

$$\bar{S}(\rho, \tau) = -\frac{1}{\rho^{1/2}} [\delta(\tau - \lambda) - \frac{5}{2} \int_0^\tau (\sqrt{\tau^2 - y^2} - \lambda) J_1(\frac{5}{2} y) dy] \dots(47)$$

where λ is given by (43)

This completes the solution of Case II.

6. SOLUTION FOR CASE III

In this case, $\alpha > 2$ and $\beta < 0$. Consequently the argument $p\rho^\beta/\beta$ of the modified Bessel functions I_n and K_n in (26) are negative for $p > 0$, so that neither function is real valued in the present case. Furthermore, in contrast to Case I, the argument now tends to zero as $\rho \rightarrow \infty$. Since $I_n(z)$ and $K_n(z)$ have linearly independent singularities at $z = 0$, the general solution (26) cannot be adopted to the regularity requirement (25) in case III. Hence the stress no longer tends to zero at all times as $\rho \rightarrow \infty$. Therefore we are compelled to abandon the regularity condition. This leaves us with a single boundary condition (24) which is sufficient to determine uniquely the values of both $A_1(p)$ and $A_2(p)$ in (26). One finds however that unless $A_1 = 0$ (26) fails to give rise to a diverging wave in the physical domain. On these grounds the functions of the first kind are inadmissible in (26).

Setting $A_1 = 0$ and proceeding as in Case I we are led formally once again to the integral equation (36) and (41) giving displacement and stresses.

7. A CLASS OF ELEMENTARY SOLUTIONS IN CLOSED FORM

Let $\alpha \neq Z$ (Case I and Case III) so that (29) represents the Laplace transform of the displacement. Moreover we recall that when n is half an odd integer, that is,

$$n = k + \frac{1}{2}, (k = 0, \pm 1, \pm 2, \dots; \dots) \quad \dots(48)$$

the modified Bessel function of the second kind degenerates into elementary functions, Watson⁶, as

$$K_{k+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{m=0}^k \frac{(k+m)!}{m! (k-m)! (2z)^m} \quad (k \geq 0) \quad \dots(49)$$

$$K_n(z) = K_n(z), \quad n \text{ arbitrary.} \quad \dots(50)$$

Thus if n in (29) obeys (49), the determination of $\bar{u}(\rho, \tau)$ from $U(\rho, p)$ reduces to the determination of the inverse of the rational function and may be carried out in closed,

form in terms of elementary functions. Consequently $\bar{S}(\rho, \tau)$ which follows from $\bar{u}(\rho, \tau)$ admits similar representation.

In view of (48) and (27) the infinite aggregate of elementary closed solutions here referred to, corresponds to the sequence of α ,

$$\alpha = \frac{4(k-1)}{3+2k}, \quad (k = 0, \pm 1, \pm 2, \dots, \dots). \quad \dots(51)$$

For convenience we give the following table which displays the initial members of the sequence (51).

k	0	1	-1	2	-2	3	-3	4	-4
α	-4/3	0	-8	4/7	12	8/9	16/3	12/11	4

8. AN EXAMPLE

We now present explicitly the closed form solutions appropriate to $\alpha = -4/3$ in Figs. 1 - 3, which belong to the case I. Since the underlying computations are elementary we give below the final results without intermediate details. Solution for $\alpha = -4/3$:

$$\bar{u}(\rho, \tau) = h(\tau - \zeta) \rho^{-2/3} e^{-5/3(\tau - \zeta)}$$

$$\bar{S}(\rho, \tau) = -\rho^{-4/3} \delta(\tau - \zeta) + \frac{5}{8} (\rho^{-4/3} - \rho^{-2}) h(\tau - \zeta) e^{-5/3(\tau - \zeta)}$$

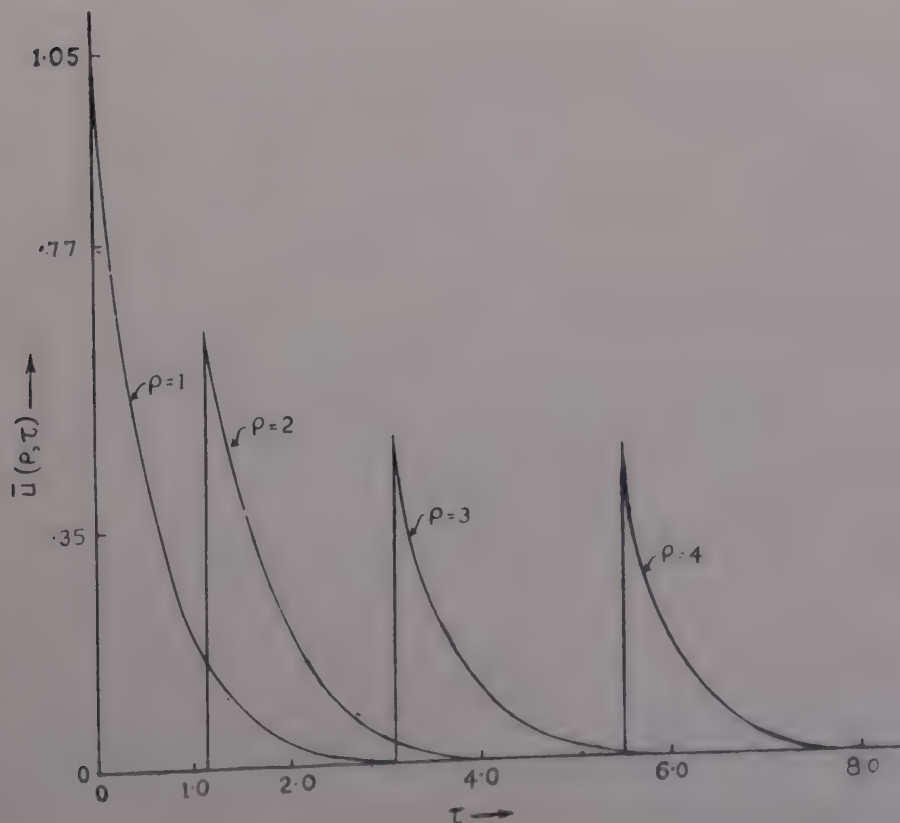


FIG. 1. $\alpha = -4/3$. Displacement \bar{u} at various positions.

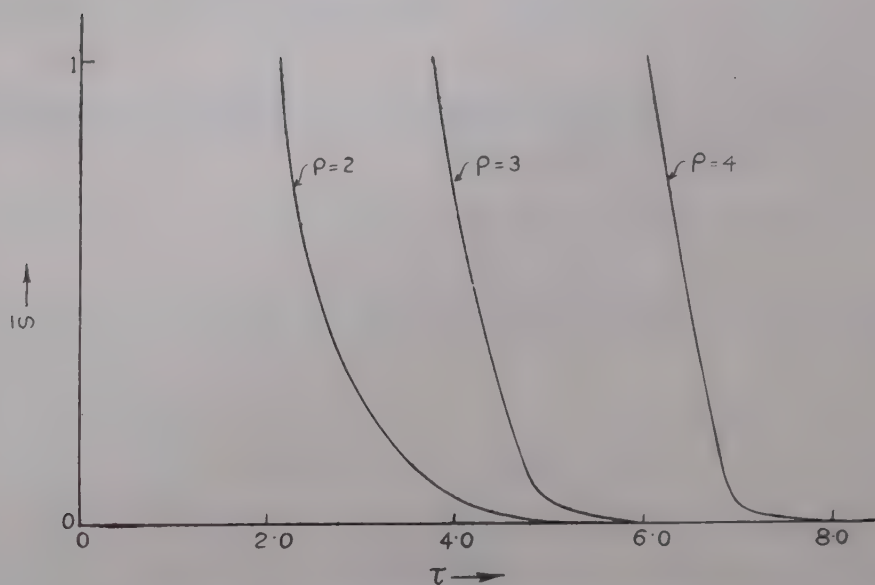


FIG. 2. $\alpha = -\frac{4}{3}$. Stress at various positions.

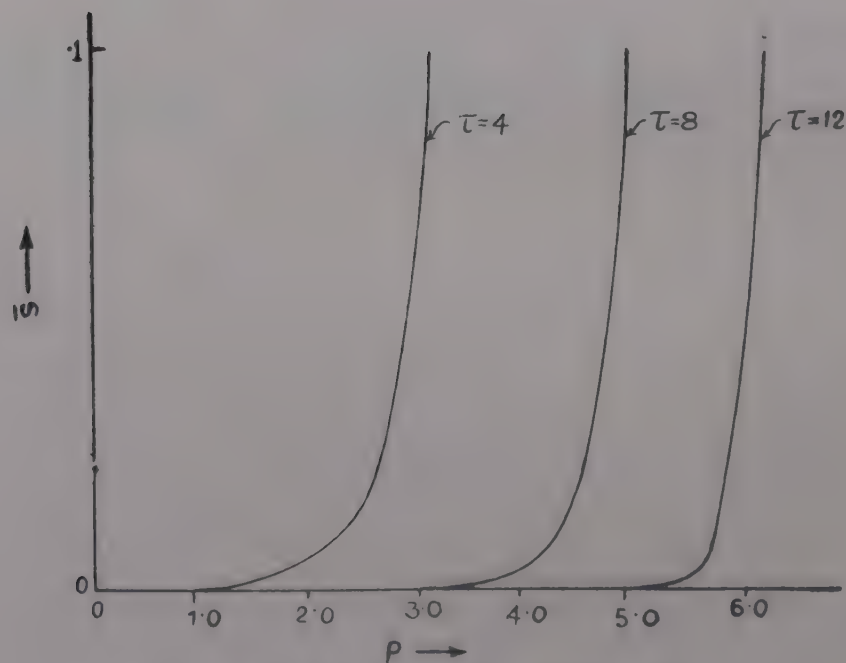
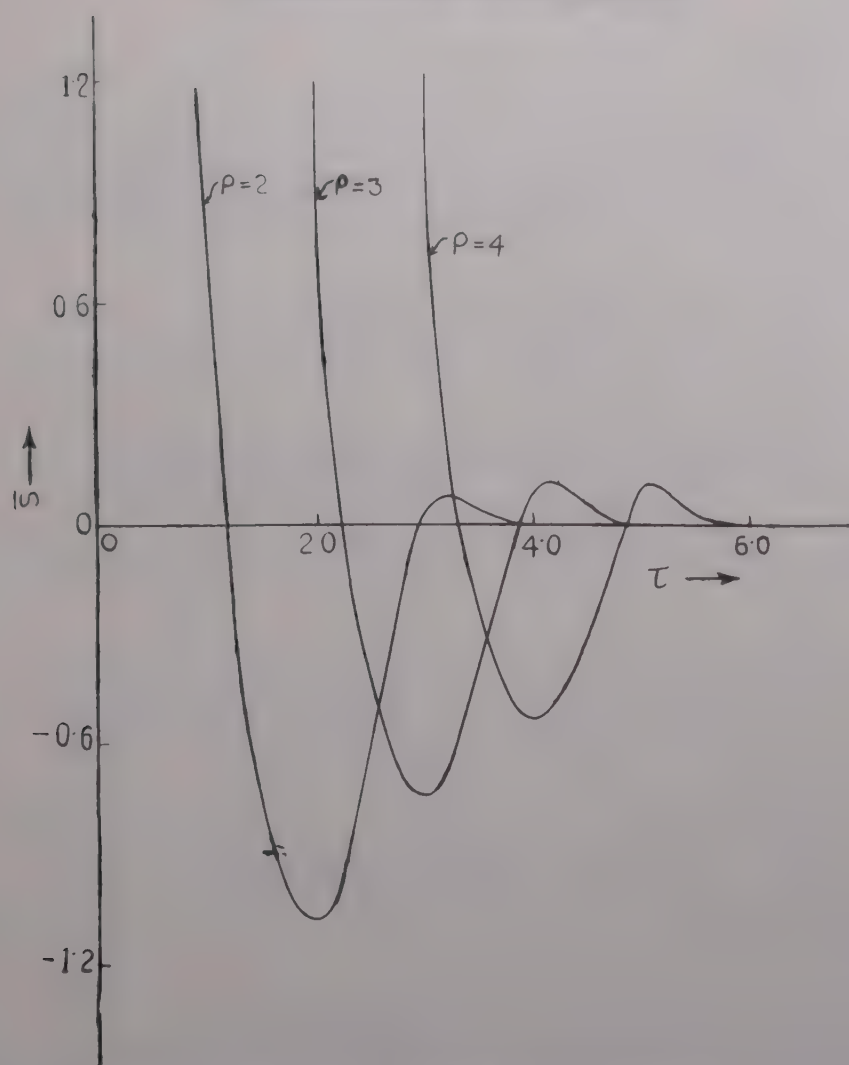


FIG. 3. $\alpha = -\frac{4}{3}$. Stress at various times.

where

$$\zeta = \frac{5}{3} (\rho^{5/3} - 1).$$

The corresponding results for the homogeneous case ($\alpha = 0$) are given below from which the figures are exhibited in Figs. 4 and 5.

FIG. 4. $\alpha = 0$. Stress at various position.

$$\begin{aligned} \bar{S}(\rho, \tau) = & -\frac{1}{\rho} \delta(\tau - m) \\ & + \frac{3m}{\rho^2} e^{-3/2(\tau-m)} h(\tau - m) \left[\cos \left\{ \frac{\sqrt{3}}{2} (\tau - m) \right\} \right. \\ & \left. + \frac{2}{\sqrt{3}} \xi \sin \left\{ \frac{\sqrt{3}}{2} (\tau - m) \right\} \right] \end{aligned}$$

where

$$m = \rho - 1, q = \frac{2 - \rho}{2\rho}.$$

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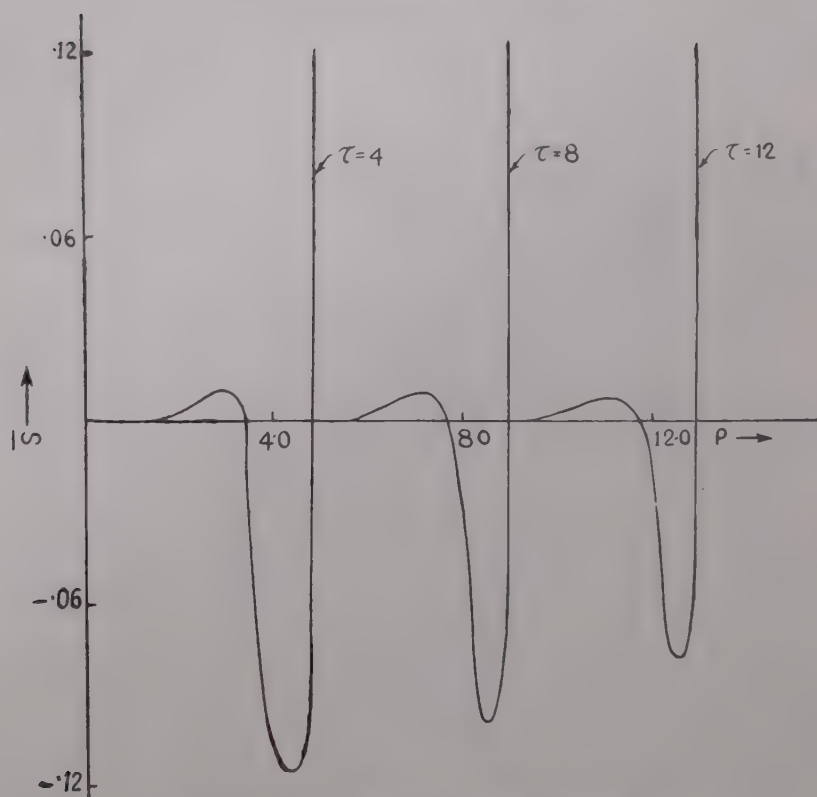


FIG. 5. $\alpha = 0$. Stress at various time.

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CORRECTIONS TO "THE NEIGHBOURHOOD NUMBER OF A GRAPH"
[INDIAN J. PURE AND APPLIED MATH., 16 (1985), 126-132] BY
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We have noted and corrected some flaws in Sampathkumar and Neeralagi's^{2,3} papers.

In this paper we consider only finite undirected graphs without loops and multiple lines. We follow the notation and terminology of Harary¹.

For a graph $G = G(V, E)$, we write $N(u) = \{v \in V : uv \in E\}$ and $N[u] = \{u\} \cup N(u)$. A set $X \subset V$ is called a neighbourhood set if $G = \bigcup_{u \in X} \langle N[u] \rangle$ where $\langle N[u] \rangle$ is the graph induced by $N[u]$. A set $T \subset E$ is called a line neighbourhood set of G if $G = \bigcup_{x \in T} \langle N[x] \rangle$ where for $x = uv$, $N[x] = N(u) \cup N(v)$. A set $X \subset V$ ($T \subset E$) is called a point cover (Line cover) for G if it covers all the lines (points) of G . A set $D \subset V$ is called a point dominating set of G if every $v \in V - D$ is adjacent to a point in D . A set $T \subset E$ is called a line dominating set of G if every $x \in E - T$ is adjacent to a line in T . A set $X \subset V$ is independent if no two points in X are adjacent. The minimum cardinalities of a neighbourhood set, a line neighbourhood set, a point covering set, a line covering set, a point dominating set and a line dominating set are called the neighbourhood number $n_0(G)$, the line neighbourhood number $n'_0(G)$, the point covering number $\alpha_0(G)$, the line covering number $\alpha_1(G)$, the point dominating number $\gamma(G)$ and the line dominating number $\gamma'(G)$ respectively. The maximum cardinality of an independent set of points is known as the point independence number $\beta_0(G)$. We shall denote the maximum degree of a vertex in G by Δ , the greatest integer $\leq p/2$ by $\lfloor p/2 \rfloor$ and the smallest integer $\geq p/2$ by $\lceil p/2 \rceil$.

Sampathkumar and Neeralagi² have stated that $n_0(G) \leq \alpha_1(G)$. This result is not correct as can be seen by considering the Petersen graph for with $n_0 = 6$ and $\alpha_1 = 5$. We give an upper bound for n_0 using α_1 as follows.

Proposition—For a (p, q) graph G ,

$$n_0 \leq \alpha_1 + k \quad \dots(1)$$

where

$$k = \min (\lfloor p/2 \rfloor - \Delta, \lfloor p/2 \rfloor - \beta_0).$$

PROOF: From the relation³ $n_0 \leq \alpha_0$, since $\alpha_0 + \beta_0 = p$ and $\alpha_1 > \lceil p/2 \rceil$ it follows that

$$n_0 \leq \alpha_1 + \lfloor p/2 \rfloor - \beta_0. \quad \dots(2)$$

Similarly, since² $n_0 \leq p - \Delta$, we get

$$n_0 \leq \alpha_1 + \lfloor p/2 \rfloor - \Delta \quad \dots(3)$$

the result follows from (2) and (3).

Note that k may be negative. It may also be mentioned that $n_0 > \alpha_1$ if G is without triangles and $\alpha_0 > \alpha_1$. This is a simple consequence of the fact that $n_0 = \alpha_0$ for a graph without triangles².

As the relation $n_0 \leq \alpha_1$ does not hold, the relation (9) and Corollary 7.1 of Sampathkumar and Neeralagi³ need corrections. The correct form of (9), in view (1), is

$$\gamma/2 \leq n'_0 \leq n_0 \leq \min(\alpha_0, \alpha_1 + k). \quad \dots(4)$$

Corollary 7.1 should read as follows :

Corollary—For a connected (p, q) graph G ,

$$\gamma' \leq \lceil q/2 \rceil. \quad \dots(5)$$

PROOF: As³ $\gamma' \leq p/2$ and is an integer, $\gamma' \leq \lfloor p/2 \rfloor$. If $q \geq p$ the result is obvious. If $q < p$ we must have $q = p - 1$ as G is connected. Hence $\gamma' \leq \lfloor p/2 \rfloor = \lceil q/2 \rceil$.

Sampathkumar and Neeralagi³ make the statement that there is no relation between γ' and n_0 , probably in the sense that no relation of the form $\gamma' \geq n_0$ or $\gamma' \leq n_0$ is true in general. However the following proposition relates n_0 and γ' .

Proposition—For any graph G ,

$$n_0 \leq 2 n'_2 \leq 2\gamma'. \quad \dots(6)$$

PROOF: Let $T = \{x_1, x_2, \dots, x_r\}$ be a minimum line neighbourhood set of G and $x_i = u_i v_i$, $1 \leq i \leq r$. Let $H = \bigcup_{i=1}^r \{u_i, v_i\}$. Then clearly H is a neighbourhood set of G . Hence $n_0 \leq |H| \leq 2|T| \leq 2n'_0$. Since³ $n'_0 \leq \gamma'$ the result follows.

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Volume 19

CONTENTS

	<i>Page</i>
A fixed point theorem for a sequence of mappings <i>by</i> R. VASUKI ...	827
On an error term related to the greatest divisor of n , which is prime to k <i>by</i> S. D. ADHIKARI, R. BALASUBRAMANIAN and A. SANKARANARAYANAN ...	830
A note on N -groups <i>by</i> V.R. YENUMULA and S. BHAVANARI ...	842
The intrinsic Gauss, Codazzi and Ricci equations for the Berwald connection in a Finsler hypersurface <i>by</i> MASAKI FUKUI ...	846
On the matrix roots of $f(X) = A$ <i>by</i> EUGENE SPIEGEL ...	854
Convex hulls and extreme points of some families of multivalent functions <i>by</i> A. K. MISHRA and MRS. P. SAHU ...	865
On linear combinations of n analytic functions in generalized Pinchuk and generalized Moulis classes <i>by</i> S. BHARGAVA and S. NANJUNDA RAO ...	875
Convolutions of certain classes of univalent functions with negative coefficients <i>by</i> K. S. PADMANABHAN and M. S. GANESAN ...	880
On the restricted problem of three rigid bodies <i>by</i> S. M. ELSHABOURY and S. ABDEL FATTAH ...	890
A note on the minimum potential strength <i>by</i> B. G. SIDHARTH and M. MAQBOOL ...	896
On the propagation of waves in a non-homogeneous fluid <i>by</i> FAWZY SHABAN EL-DEWIK ...	901
Propagation characteristics in distensible tubes containing a dusty viscous fluid <i>by</i> S. P. SINGH G. C. CHADDA and A. K. SINHA ...	906
Disturbance in a non-homogeneous elastic medium by a twisting impulsive force <i>by</i> NANDINI CHAKRAVORTY ...	915
Corrections to "The neighbourhood number of a graph" <i>by</i> P. P. KALL and N. V. DESHPANDE ...	927